Classifying Model-Theoretic Properties

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Background

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- 1965: Morley proves categoricity theorem (beginning of modern model theory).
- 1970s,1980s: Computable model theory.

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Do decidable Vaughtian models always exist for a given complete decidable theory?

Negative Results:

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Theorem (Millar, Goncharov, Peretyat'kin)

There is a homogeneous model with a uniformly computable list of types, but with no decidable copy.

Positive Results:

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If $\mathcal{A}$ is prime and has a $\mathbf{0}$-basis, then $\mathcal{A}$ has a decidable presentation.

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If $\mathcal{A}$ is prime and has a $\mathbf{0}$-basis, then $\mathcal{A}$ has a decidable presentation.

Theorem (Millar, Morley)

If $\mathcal{A}$ is saturated and has a $\mathbf{0}$-basis, then $\mathcal{A}$ has a decidable presentation.

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Theorem (Csima)

For any CAD theory T, there is a prime model of T decidable in some low degree.

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Theorem (Csima, Hirschfeldt, Knight, Soare)

A Turing degree $\mathbf{d} \in \Delta^0_2$ *is prime bounding if and only if it is nonlow*$_2$*; i.e.* $\mathbf{0}'' <_T \mathbf{d}''$.

The Nine Predicates

CHKS introduced nine predicates of a degree $\mathbf{d}$, all of which are equivalent if $\mathbf{d} \leq \mathbf{0}'$.

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(P0) *Prime bounding.* For any CAD theory T, there is a prime model of T decidable in $\mathbf{d}$.

(P1) *Isolated path predicate.* For any computable tree $T \subseteq 2^{<\omega}$ with no terminal nodes and isolated paths dense, there is a function $g(\sigma, t) \leq_T \mathbf{d}$ such that for every fixed $\sigma \in T$, $g(\sigma, t) = g_\sigma \in 2^\omega$ is an isolated path in T extending σ.

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(P2) *Escape predicate.* For any given function $f \leq_T \mathbf{0}'$, there is a function $g \leq_T \mathbf{d}$ such that for infinitely many $x \in \omega$ we have

$$f(x) \leq g(x).$$

(P3) *Equivalence structure predicate.* For any infinite Δ^0_2 set $S \subseteq \omega \setminus \{0\}$, there is a **d**-computable equivalence structure with one class of size n for each $n \in S$, and no other classes.

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(P4) *Nonlow$_2$.* $\mathbf{0}'' <_T \mathbf{d}''$.

New Results

Theorem (Conidis, Csima, Hirschfeldt, Knight, Soare)

The nine predicates of [CHKS] fall into three equivalence classes under implication. One of size 5, one of size 3, and one of size 1. Furthermore, the class of size 5 implies the class of size three, and no other implications exist amongst the predicates.

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Corollary (Conidis, Csima, Hirschfeldt, Knight, Soare)

$$[(\mathrm{P0}) \Leftrightarrow (\mathrm{P1}) \Leftrightarrow (\mathrm{P2})] \Rightarrow [(\mathrm{P3})]$$

$$[(\mathrm{P4})]$$

(P0) implies (P2)

Theorem (Conidis)

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$$(\mathrm{P0}) \Leftrightarrow (\mathrm{P1}) \Leftrightarrow (\Pi^0_1 - \mathrm{P1})$$

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Main idea: Given $f \leq_T \mathbf{0}'$, construct a computable tree $T \subseteq 2^{<\omega}$ with no terminal nodes and isolated paths dense, such that the isolated paths of T code infinitely many values $\langle x, f(x)\rangle, x \in \omega$.

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Main idea of proof: Construct a tree $T \subset 2^{<\omega}$ such that every path through T does not satisfy the prime bounding predicate. Using $\mathbf{0}''$, find a path through T that satisfies the equivalence structure predicate.

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The nonlow$_2$ predicate does not imply the equivalence structure predicate.

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The nonlow$_2$ predicate does not imply the prime bounding predicate.

Main idea of proof: Construct a perfect tree $T \subset 2^{<\omega}$ such that every path through T does not satisfy the equivalence structure predicate. Tree version of the proof that $\mathbf{0}$ does not satisfy the equivalence structure predicate.

0 does not satisfy (P3)

Definition

A Turing degree **d** satisfies the *monotone predicate* if for every infinite $S \in \Delta^0_2$ there is a function $f(x, y) \leq_T \mathbf{d}$ such that:

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- $f(x, 0) = x$, for every $x \in \omega$.

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The monotone predicate is equivalent to the equivalence structure predicate (P3) [CHKS].

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- Want to construct an infinite set $S \in \Delta^0_2$ such that for every $e \in \omega$, if $(\forall x)\varphi_e(x,0) = x$ and $\varphi_e(x,y)$ is nondecreasing in y, then there is some $x_e \in \omega$ such that $\lim_y \varphi_e(x_e,y) \notin S$ or $\lim_y \varphi_e(x_e,y) = \infty$.

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- Build $S \subset \omega$ in stages.

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- Build $S \subset \omega$ in stages.
- Stage s: For every $0 \leq t \leq s$, ask $\mathbf{0}'$ whether

$$(\exists y)[\varphi_t(c(t), y) > c(s + 1)].$$

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- Build $S \subset \omega$ in stages.
- Stage s: For every $0 \leq t \leq s$, ask $\mathbf{0}'$ whether

$$(\exists y)[\varphi_t(c(t), y) > c(s+1)].$$

- Then, find a number $c \in [c(s), c(s+1))$ that is not a candidate for $\lim_y \varphi_t(c(t), y)$, and put c into S.

Related Results of Hirschfeldt, Shore, and Slaman

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- Showed that in the context of reverse mathematics (P1) and (P2) are not equivalent.

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- Called (P1) Atomic Model Theorem.
- Showed that in the context of reverse mathematics (P1) and (P2) are not equivalent.
- Over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$, (P1) is Π^1_1-conservative, but (P2) implies $\mathsf{I}\Sigma_2$.

Related Results of Hirschfeldt, Shore, and Slaman

Theorem (Conidis)

In every ω-model of RCA_0, the nine predicates of [CHKS] fall into three equivalence classes under implication. One of size 5, one of size 3, and one of size 1. Moreover, the class of size 5 implies the class of size 3, and no other implications exist amongst the predicates.

References

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