Completions and model completions of co-Heyting algebras

Luck Darnière Markus Junker

August 4, 2009

1 - (Co)dimension

We consider only distributive bounded lattices in the language $\mathcal{L}_{lat} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$.

For every prime filter $\mathfrak{p}$ of L let:

- height $\mathfrak{p}$ = the foundation rank of $\mathfrak{p}$
- coheight $\mathfrak{p}$ = the cofoundation rank of $\mathfrak{p}$

For every element a of L let:

- dim $a = \sup\{\text{coheight}\,\mathfrak{p} \;/\; \mathfrak{p}$ prime filter, $a \in \mathfrak{p}\}$
- codim $a = \min\{\text{height}\,\mathfrak{p} \;/\; \mathfrak{p}$ prime filter, $a \in \mathfrak{p}\}$

L is finite dimensional if every a in L has finite dimension.

Fact

$$\begin{aligned} \dim a \vee b &= \max(\dim a, \dim b) \\ \operatorname{codim} a \vee b &= \min(\operatorname{codim} a, \operatorname{codim} b) \end{aligned}$$

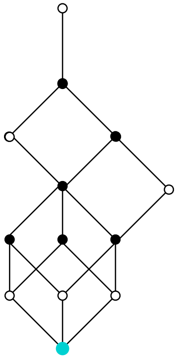


Figure: $\dim \mathbf{0} = -\infty$

Fact

$$\begin{aligned} \dim a \vee b &= \max(\dim a, \dim b) \\ \operatorname{codim} a \vee b &= \min(\operatorname{codim} a, \operatorname{codim} b) \end{aligned}$$

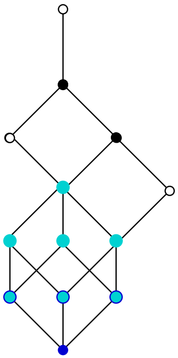


Figure: Points of dimension 0

Fact

$$\begin{aligned} \dim a \vee b &= \max(\dim a, \dim b) \\ \operatorname{codim} a \vee b &= \min(\operatorname{codim} a, \operatorname{codim} b) \end{aligned}$$

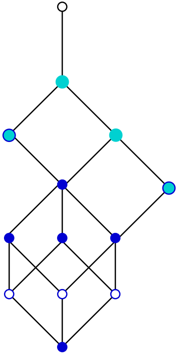


Figure: Points of dimension 1

Fact

$$\begin{aligned} \dim a \vee b &= \max(\dim a, \dim b) \\ \operatorname{codim} a \vee b &= \min(\operatorname{codim} a, \operatorname{codim} b) \end{aligned}$$

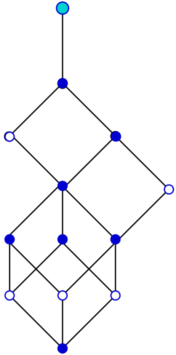


Figure: Points of dimension 2

Example

$L =$ lattice of Zariski closed subset of the affine n-space k^n over an infinite field k. For every $A \in L$, (co)dim$A$ coincides with the geometric (co)dimension of A (in k^n).
$A - B$=Zariski closure of $A \setminus B$ belongs to L.

A co-Heyting algebra is a bounded distributive lattice with an additional binary operation $a - b = \min\{c \ / \ a \leq b \vee c\}$.

Lemma

Let d be a positive integer. There are positive existential formulas ϕ_d, ψ_d in the language of co-Heyting algebras, such that for every co-Heyting algebra L and every $a \in L$:

$$\dim a \geq d \iff L \models \varphi_d(a)$$

$$\operatorname{codim} a \geq d \iff L \models \psi_d(a)$$

2 - Completion

In every co-Heyting algebra L define:

$$\delta(a, b) = 2^{-\operatorname{codim}_L (a-b)\vee(b-a)}$$

Fact (Triangle ultrametric inequality)

$$\delta(a, c) \leq \max \delta(a, b), \delta(b, c)$$

δ is a pseudo-metric, hence defines a topology on L. The operations $\vee, \wedge, -$ are uniformly continuous for δ
δ is an ultrametric iff its topology is separated. In this case we say that L is separated.

The completion $\widehat{L}$ of L with respect to δ is the set of equivalence classes of Cauchy sequences. The algebraic structure of L extends uniquely to $\widehat{L}$ by uniform continuity.

Theorem

The completion of L is also the projective limit of all its finite dimensional quotients.

Remark

$dL = \{a \,/\, \operatorname{codim} a \geq d\}$ is an ideal of L, for every positive integer d. The quotients L/dL form a projective system, and:

$$\widehat{L} \simeq \varprojlim (L/dL)_{d<\omega}$$

Theorem

The completion of L is also the projective limit of all its finite dimensional quotients.

Remark

$dL = \{a /\ \operatorname{codim} a \geq d\}$ is an ideal of L, for every positive integer d. The quotients L/dL form a projective system, and:

$$\widehat{L} \simeq \varprojlim (L/dL)_{d<\omega}$$

Corollary

Every monotonic sequence in a compact subset of $\widehat{L}$ converges.

Corollary

If $\widehat{L}$ is compact, then it is bi-Heyting.

3 - Precompactness

A co-Heyting algebra L is precompact if its completion is compact.

Theorem

In every variety $\mathcal{V}$ of co-Heyting algebras, the following are equivalent:

1. *$\mathcal{V}$ has the finite model property.*
2. *Every algebra free in $\mathcal{V}$ is Hausdorff.*
3. *Every algebra finitely presented in $\mathcal{V}$ is precompact Hausdorff.*

Corollary

Every finitely presented co-Heyting algebra is precompact Hausdorff.

Remark

The class of precompact Hausdorff co-Heyting algebra is much larger than the class of finitely presented ones.

Theorem

Let L be a precompact Hausdorff co-Heyting algebra.

1. **L and $\widehat{L}$ have the same completely join (resp. meet) irreducible elements.**

Remark

The class of precompact Hausdorff co-Heyting algebra is much larger than the class of finitely presented ones.

Theorem

Let L be a precompact Hausdorff co-Heyting algebra.

1. *L and $\widehat{L}$ have the same completely join (resp. meet) irreducible elements.*
2. **Every join irreducible element of L is completely join irreducible.**

Remark

The class of precompact Hausdorff co-Heyting algebra is much larger than the class of finitely presented ones.

Theorem

Let L be a precompact Hausdorff co-Heyting algebra.

1. *L and $\widehat{L}$ have the same completely join (resp. meet) irreducible elements.*
2. *Every join irreducible element of L is completely join irreducible.*
3. **Every element of L is the complete meet of all the completely meet irreducible elements greater than it.**

Remark

The class of precompact Hausdorff co-Heyting algebra is much larger than the class of finitely presented ones.

Theorem

Let L be a precompact Hausdorff co-Heyting algebra.

1. *L and $\widehat{L}$ have the same completely join (resp. meet) irreducible elements.*
2. *Every join irreducible element of L is completely join irreducible.*
3. *Every element of L is the complete meet of all the completely meet irreducible elements greater than it.*
4. **Every element of L is the complete join of its join irreducible components.**

Let L be a precompact Hausdorff co-Heyting algebra.

Question 1

Is L existentially closed in its completion?

Question 2

Is L an elementary substructure of $\widehat{L}$?

Let L be a precompact Hausdorff co-Heyting algebra.

Question 1

Is L existentially closed in its completion?

Question 2

Is L an elementary substructure of $\widehat{L}$?

Theorem

Let $\mathcal{F}_n$ be the free co-Heyting algebra with n generators.
If $\mathcal{F}_n \equiv \widehat{\mathcal{F}}_n$ then $\mathcal{F}_n \preccurlyeq \widehat{\mathcal{F}}_n$

Ingredient of the proof: $\mathcal{F}_n$ has only one set of free generators, definable in $\mathcal{F}_n$ and $\widehat{\mathcal{F}}_n$ by the same formula.

4- Density and splitting

The strong order $\ll$ is defined on L by:

$$b \ll a \iff b \leq a \text{ and } a - b = a$$

Example

$L =$ the co-Heyting algebra of Zariski closed subsets of k^n.
$B \ll A$ iff B has empty interior in A.

Fact

In every co-Heyting algebra L, $\dim a$ is the foundation rank of a in $L \setminus \{\mathbf{0}\}$ with respect to $\ll$.

We introduce now two axioms of co-Heyting algebras.

- Density (D1)
  If $c \ll a \neq \mathbf{0}$ then there exists a non zero element b such that:
  $$c \ll b \ll a$$
- Splitting (S1)
  If $b_1 \vee b_2 \ll a \neq \mathbf{0}$ then their exists non zero elements a_1 and a_2 such that:
  $$a - a_2 = a_1 \geq b_1$$
  $$a - a_1 = a_2 \geq b_2$$
  $$a_1 \wedge a_2 = b_1 \wedge b_2$$

Remark

Since $a_1 \vee a_2 = (a - a_2) \vee a_2 = a$ the second axiom allows to split a in two pieces a_1, a_1 along b_1, b_2 (so the name).

$L =$ the lattice of closed semi-algebraic subsets of $\mathbb{R}^2$.

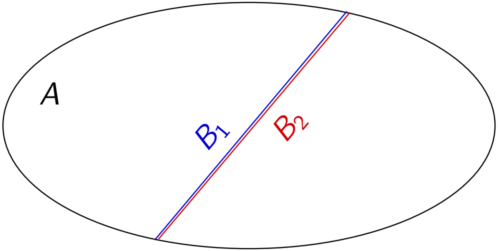


Figure: Splitting of an ellipse A along $B_1 = B_2$

Remark

Since $a_1 \vee a_2 = (a - a_2) \vee a_2 = a$ the second axiom allows to split a in two pieces a_1, a_1 along b_1, b_2 (so the name).

$L =$ the lattice of closed semi-algebraic subsets of $\mathbb{R}^2$.

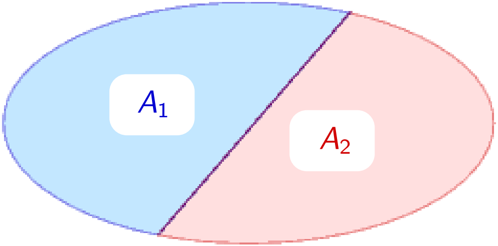


Figure: Splitting of an ellipse A along $B_1 = B_2$

Remark

In order to split A along B_1, B_2 in L, it is necessary (not sufficient) that $A \setminus (B_1 \cup B_2)$ is not connected.

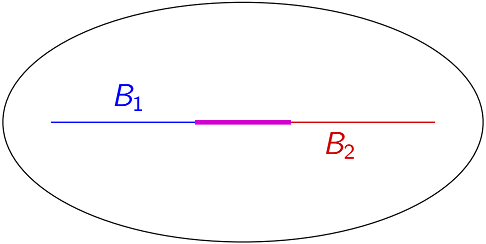
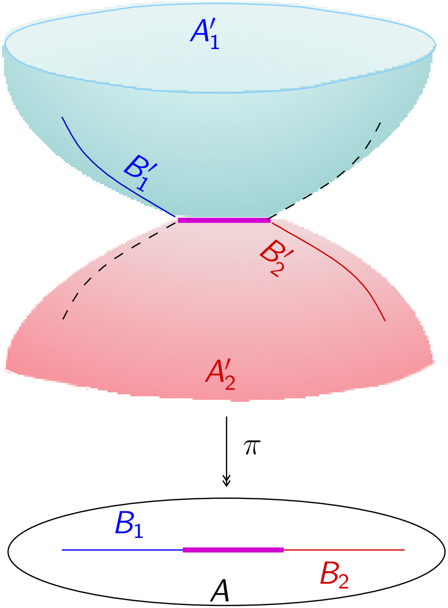


Figure: No splitting... in L


A'_1
B'_1
B'_2
A'_2
π
B_1
B_2
A

The map $f : A \in L \mapsto \pi^{-1}(A)$ embeds L into a co-Heyting algebra L' in which the image of A can be splited along the images B'_1, B'_1 of B_1, B_2:

$$f(A) = \pi^{-1}(A) = A'_1 \cup A'_2$$

Theorem

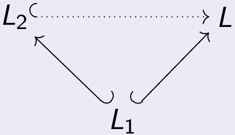
Every co-Heyting algebra embeds into a co-Heyting algebra satisfying the density and splitting axioms D1, S1.

Corollary

Every existentially closed co-Heyting algebra satisfies the density and splitting axioms D1, S1.

Theorem

Let L_1, L_2, L be co-Heyting algebras. If L_2 is finite and L satisfies axioms D1 and S1 then every embedding of L_1 into L extends to an embedding of L_2 into L.



Question 3

If L_1, L_2 are finitely generated and L satisfies axioms D1 and S1, does every embedding of L_1 into L extend to an embedding of L_2 into an elementary extension of L?

5- Model completion

Theorem (L. Maksimova)

There are exactly eight varieties of co-Heyting algebras having the amalgamation property.

Remark

- Only the theories $T_1, \ldots, T_8$ of these varieties can have a model-completion.
- We can forget about T_8 (theory of the one-point co-Heyting algebra) and T_7 (theory of boolean algebras) whose model-theoretic properties are well know.

Theorem (A. Pitts)

The second order intuitionistic propositional calculus is interpretable in the first order one.

Theorem (A. Pitts, S. Ghilardi, M. Zawadowski)

Each theory $T_1, \ldots, T_6$ has a model-completion.

Ingredients of the proof:

- The amalgamation property for $T_1, \ldots, T_6$.
- Pitts's theorem for T_1 (the theory of all co-Heyting algebras), and an adaptation of it for T_2.
- General model-theoretic non-sense for $T_3, \ldots, T_6$, using that these theories are locally finite.

Remark

No meaningful axiomatization of these model-completions is given by this approach.

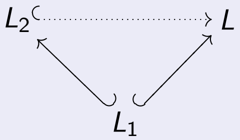
For each k between 1 and 6, we introduced two axioms Dk, Sk adapting to T_k the density and splitting axioms D1, S1 of T_1.

Theorem

Every model of T_k embeds into a model of T_k satisfying the density and splitting axioms Dk and Sk.

Theorem

Let L_1, L_2, L be models of T_k. If L_2 is finite and L satisfies axioms Dk and Sk then every embedding of L_1 into L extends to an embedding of L_2 into L.



Since every finitely generated model of T_k is finite for $k = 3, 4, 5, 6$ it follows immediately that:

Corollary

For $k = 3, 4, 5, 6$ the theory T_k has a model-completion which is axiomatized by the density and splitting axioms Dk, Sk.

Since every finitely generated model of T_k is finite for $k = 3, 4, 5, 6$ it follows immediately that:

Corollary

For $k = 3, 4, 5, 6$ the theory T_k has a model-completion which is axiomatized by the density and splitting axioms Dk, Sk.

Let $\mathcal{L}_k$ denote the superintuitionistic logic corresponding to the variety of Heyting algebras whose duals are models of T_k.

Corollary

The second order propositional calculus of $\mathcal{L}_k$ is interpretable in the first order one.

References

- Codimension and pseudometric in co-Heyting algebras (arXiv:0812.2026) *Submitted*
- On Bellissima's construction of the finitely generated free Heyting algebras, and beyond (arXiv:0812.2027) *Submitted*
- **Model completion of equational theories of co-Heyting algebras** (Soon on Arxiv) *Looking for a journal*