Splitting and Antisplitting Theorems in Classes of Low Degrees

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Turing Jumps in the Ershov Hierarchy

A Turing degree **a** $= deg(A)$ is called *superlow* if $A' \in \Delta^{-1}_{\omega}$ (or, equivalently, $A' \leqslant_{\mathrm{tt}} \emptyset'$).

Theorem 1. The set A is superlow if and only if $A' \in \Delta^{-1}_{\omega\times n}$ for some n.

Theorem 2. More generally, the proper levels for Turing jumps in the Ershov hierarchy for the natural notation system are Σ_1^{-1} and $\Delta_{\omega^n}^{-1}$, $n > 0$.

Theorem 3. For each $n > 0$ the class

$$\{A : A' \in \Delta_{\omega^n}^{-1}\}$$

(and in particular the class of all superlow sets) has a Δ_2^0-computable numbering.

A Turing degree $\mathbf{a}$ is called *totally ω-c.e.* if every function $g \leqslant_T \mathbf{a}$ is ω-c.e. It is known that each superlow c.e. degree is totally ω-c.e.

Three incomparable elements $\mathbf{a}_0$, $\mathbf{a}_1$ and $\mathbf{b}$ in an upper semilattice form a *weak critical triple* if $\mathbf{a}_0 \cup \mathbf{b} = \mathbf{a}_1 \cup \mathbf{b}$ and there is no $\mathbf{c} \leqslant \mathbf{a}_0, \mathbf{a}_1$ with $\mathbf{a}_0 \leqslant \mathbf{b} \cup \mathbf{c}$. Downey, Greenberg and Weber (2007) have proved that a degree $\mathbf{a}$ is totally ω-c.e. if and only if $\mathbf{a}$ does not bound a weak critical triple.

The Joins of Superlow c.e. Degrees

Let $\mathbf{C}$ be the semilattice of all c.e. degrees, and let

$$\mathbf{J} = \{\mathbf{a}\cup\mathbf{b} : \mathbf{a}, \mathbf{b} \text{ are superlow c.e. degrees}\}.$$

Bickford and Mills (1982) have proved, that $\mathbf{0}' \in \mathbf{J}$.

Theorem 4. For all superlow c.e. degrees $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ there are superlow c.e. degrees $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, such that

$$\mathbf{a}_0 \cup \mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{b}_0 \cup \mathbf{b}_1 = \mathbf{b}_0 \cup \mathbf{b}_2 = \mathbf{b}_1 \cup \mathbf{b}_2.$$

Corollary 5. $\mathbf{J}$ is an upper semilattice.

Theorem 6. There is a c.e. degree $\mathbf{a}$ such that for all c.e. degrees $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$ if

$$\mathbf{a} = \mathbf{b}_0 \cup \mathbf{b}_1 = \mathbf{b}_0 \cup \mathbf{b}_2 = \mathbf{b}_1 \cup \mathbf{b}_2$$

then $\mathbf{b}_i$ is not totally ω-c.e. for some $i < 3$.

Corollary 7. The upper semilattices $\mathbf{C}$ and $\mathbf{J}$ are not elementary equivalent.

Proof. Let Φ be the following formula:

$\forall \mathbf{x} \exists \mathbf{y}_0 \exists \mathbf{y}_1 \exists \mathbf{y}_2 [\mathbf{x} = \mathbf{y}_0 \cup \mathbf{y}_1 = \mathbf{y}_0 \cup \mathbf{y}_2 = \mathbf{y}_1 \cup \mathbf{y}_2 \,\&$

$\mathbf{y}_i$ does not bound a weak critical triple, $i < 3]$.

$$\mathbf{C} \models \neg\Phi,$$

$$\mathbf{J} \models \Phi.$$

The low c.e. degrees and the low 2-c.e. degrees are not elementary equivalent.

In the the following theorem the class Δ_a^{-1} is the Δ-level of the Ershov Hierarchy corresponding to the notation $a \in O$.

Theorem 8. For all notation $a \in O$ there is a low 2-c.e. set D such that for all 2-c.e. sets E and F if $E' \in \Delta_a^{-1}$, $F' \in \Delta_a^{-1}$ then $D \not\equiv_T E \oplus F$.

Corollary 9. (Independently with M. Yamaleev). The low c.e. degrees and the low 2-c.e. degrees are not elementary equivalent.

Proof. Let Φ be the following formula:

$$\exists \mathbf{x}_0 \exists \mathbf{x}_1 \forall \mathbf{y} \exists \mathbf{y}_0 \leqslant \mathbf{x}_0 \exists \mathbf{y}_1 \leqslant \mathbf{x}_1 [\mathbf{y} = \mathbf{y}_0 \cup \mathbf{y}_1].$$

$$D_1^{low} \models \Phi,$$

$$D_2^{low} \models \neg \Phi.$$