Bounding, splitting and almost disjoint families

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Definition

A family $\mathcal{H} \subseteq {}^\omega\omega$ is unbounded, if there is no $g \in {}^\omega\omega$ which dominates all elements of $\mathcal{H}$. The bounding number $\mathfrak{b}$ is the minimal cardinality of an unbounded family.

Definition

A family $S \subseteq [\omega]^\omega$ is splitting, if for every $A \in [\omega]^\omega$ there is $B \in S$ such that both $A \cap B$ and $A \cap B^c$ are infinite. The splitting number $\mathfrak{s}$ is the minimal cardinality of a splitting family.

Definition

A family $\mathcal{A} \subseteq [\omega]^\omega$ is maximal almost disjoint if all distinct elements of $\mathcal{A}$ have finite intersection and for every $C \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that $|A \cap C| = \omega$. The maximal almost disjointness number $\mathfrak{a}$ is the minimal size of a maximal almost disjoint family.

The bounding number is less or equal the almost disjointness number.

The bounding and the splitting numbers are independent.

- In 1985 J. Baumgartner and P. Dordal showed that $\mathfrak{s} < \mathfrak{b}$ in the Hechler model.
- In 1984 S. Shelah showed the consistency of $\mathfrak{b} < \mathfrak{s}$ using countable support iteration of proper forcing posets.

Theorem (S. Shelah, 1984)

(CH) There is a proper forcing notion Q of size $\mathfrak{c}$ which is almost ${}^{\omega}\omega$-bounding and adds a real not split by the ground model reals.

Thus under an $\aleph_2$-length iteration of Q one obtains the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. Similar arguments give the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \mathfrak{a} = \omega_2$, $\mathfrak{b} = \mathfrak{a} = \omega_1 < \mathfrak{s} = \omega_2$.

Definition (S. Shelah, 1984)

Let $\mathcal{P} \subseteq [\omega]^{<\omega}$ be an upwards closed family, which does not contain singletons. Then $\mathcal{P}$ inductively induces a function $h : [\omega]^{<\omega} \to \omega$, called a logarithmic measure, as follows:

- $h(e) > 0$ if and only if $e \in \mathcal{P}$
- for every $\ell \geq 1$ and $e \in [\omega]^{<\omega}$, $h(e) \geq \ell + 1$ if and only if for all e_0, e_1 such that $e = e_0 \cup e_1$ either $h(e_0) \geq \ell$ or $h(e_1) \geq \ell$.

For every $e \in [\omega]^{<\omega}$ let $h(e) = \max\{\ell : h(e) \geq \ell\}$.

Example

Let $\mathcal{P} = \{a \in [\omega]^{<\omega} : |a| \geq 2\}$ and let h be the induced logarithmic measure. Then $h(a) = \min\{j : |a| \leq 2^j\}$.

Sufficient condition for high values

Let h be an induced logarithmic measure. If for every finite partition $\omega = \bigcup_{j \in n} A_j$, there is A_j which contains a positive set, then for every $k \in \omega$ and finite partition $\omega = \bigcup_{j \in n} A_j$ there is A_j which contains a set of measure $\geq k$.

Definition (S. Shelah, 1984)

Let Q be the set of all pairs (u, T) where $u \in [\omega]^{<\omega}$ and $T = \langle (s_i, h_i) \rangle_{i\in\omega}$ is a sequence of logarithmic measures such that

1. $\max u < \min s_0$
2. $\max s_i < \min s_{i+1}$ for all $i \in \omega$
3. $\langle h_i(s_i) : i \in \omega \rangle$ is unbounded.

The sequence $T = \langle (s_i, h_i) : i \in \omega \rangle$ is called a pure condition. Let $\text{int}(T) = \bigcup_{i\in\omega} s_i$. Note that if $(u, T) \in Q$, then $(u, \text{int}(T))$ is a Mathias condition.

Definition (V.F., J. Steprāns, 2008)

Let C be a centered family of pure conditions. Then $Q(C)$ is the suborder of Q of all (u, T) such that $\exists R \in C(R \leq T)$.

- $Q(C)$ is σ-centered.
- If $C \subseteq Q(C')$, then C' is said to extend C.
- If $T \not\leq C$ and $\omega = \bigcup_{j \in n} A_j$, then $\exists j \in n\, \exists R \leq T(R \not\leq C)$ such that $\text{int}(R) \subseteq A_j$.

Theorem (V.F., J. Steprāns, 2008)

Let κ be a regular uncountable cardinal, $cov(\mathcal{M}) = \kappa$, $\mathcal{H} \subseteq {}^\omega\omega$ an unbounded, directed family of size κ. Let C be a centered family, $|C| < \kappa$ and let $\dot{f}$ be a good $Q(C)$-name for a real. Then there are a centered family C' extending C, $|C| = |C'|$ and $h \in \mathcal{H}$ such that $\Vdash_{Q(C'')} \check{h} \not\leq^* \dot{f}$, for every C'' extending C'.

Theorem (V.F., J. Steprāns, 2008)

Let κ *be a regular uncountable cardinal,* $\mathrm{cov}(\mathcal{M}) = \kappa = \mathfrak{c}$, $\mathcal{H} \subseteq {}^\omega\omega$ *an unbounded directed family of size* κ*. Then there is a centered family* C, $|C| = \kappa$ *such that* $Q(C)$ *preserves the unboundedness of* $\mathcal{H}$ *and adds a real not split by* $V \cap [\omega]^\omega$*.*

Proof

Let $\mathcal{N} = \{\dot{f}_\alpha\}_{\alpha<\kappa}$ enumerate all $Q(C')$ names for functions in ${}^\omega\omega$ where $|C'| < \kappa$. Let $\mathcal{A} = \{A_{\alpha+1}\}_{\alpha<\kappa}$ enumerate $V \cap [\omega]^\omega$. By induction of length κ obtain a sequence $\langle C_\alpha : \alpha < \kappa \rangle$ such that $\forall \alpha < \beta C_\alpha \subseteq Q(C_\beta)$, $|C_\alpha| < \kappa$ as follows:

- Begin with any T and $C_0 = \{T \backslash v : v \in [\omega]^{<\omega}\}$
- If α is a limit, let $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$

cont.

If $\alpha = \beta + 1$, let $\dot{g}_\alpha$ be the name with least index in $\mathcal{N} \backslash \{\dot{g}_{\gamma+1}\}_{\gamma<\beta}$ which is a $Q(C_\beta)$-name. Find C_α such that $|C_\alpha| = |C_\beta|$ and

- If $\dot{g}_\alpha$ is good, $\exists h_\alpha \in \mathcal{H} \forall C''$ extending $C_\alpha \Vdash_{Q(C'')}$ "$\check{h}_\alpha \not<^* \dot{g}_\alpha$"
- If $\dot{g}_\alpha$ is not good, then $\dot{g}_\alpha$ is not a $Q(C_\alpha)$-name
- $\exists T_\alpha \in Q(C_\alpha)(\text{int}(T_\alpha) \subseteq A_\alpha$ or $\text{int}(T_\alpha) \subseteq A_\alpha^c)$.

Then let $C = \bigcup_{\alpha<\kappa} C_\alpha$.

cont.

$\mathcal{H}$ is unbounded

If $\dot{f}$ is a $Q(C)$-name, then $\exists\beta\in\kappa$ such that $\dot{f}$ is a good $Q(C_\beta)$-name and is the name with least index in $\mathcal{N}\backslash\{\dot{g}_{\gamma+1}\}_{\gamma<\beta}$ which is a $Q(C_\beta)$-name. Then $(\mathcal{H}\text{ is unbounded})^{V^{Q(C)}}$.

$\exists$ a real not split by the ground model reals

Let G be $Q(C)$-generic. If $A\in V\cap[\omega]^\omega$ then $\exists(u,T)\in G$ such that $\mathrm{int}(T)\subseteq A$ or $\mathrm{int}(T)\subseteq A^c$. If $U_G=\bigcup\{u:\exists T(u,T)\in G\}$, then $U_G\subseteq^*\mathrm{int}(T)$ for all T such that $\exists u(u,T)\in G$.

□

Theorem (V.F, J. Steprāns, 2008)

(GCH) Let κ be a regular uncountable cardinal. There is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.

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Remark

It is relatively consistent that $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \kappa^+$.

Theorem (V.F., J. Steprāns, 2008)

(CH) There is countably closed, $\aleph_2$-cc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}\times\mathbb{C}(\omega_2)}$ there is a centered family C with the property that $Q(C)$ adds a real not split by $V^{\mathbb{C}(\omega_2)} \cap [\omega]^\omega$ and preserves the unboundedness of all families of Cohen reals of size ω_1.

This might be considered a first step towards the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \lambda$ for κ, λ arbitrary regular uncountable cardinals.

Theorem (J. Brendle, V.F., 2009)

Let $\kappa < \lambda$ be regular uncountable cardinals. There is a ccc generic extension in which $\mathfrak{a} = \mathfrak{b} = \kappa < \mathfrak{s} = \lambda$.

Theorem (J. Brendle, V.F., 2009)

Let μ be a measurable cardinal, $\mu < \kappa < \lambda$, κ and λ regular. There is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$.

How about three distinct cardinals?

- $\mathfrak{b} = \kappa < \mathfrak{s} = \lambda < \mathfrak{a} = \nu$
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