On Constructive Models of Theories with Linear Rudin-Keisler Ordering

Alexander N. Gavryushkin
gavryushkin@gmail.com
Novosibirsk State University

Definition

A model $\mathfrak{A}$ is said to be ***decidable*** if the set $\{\varphi(a_0, \ldots, a_n) \mid \mathfrak{A} \models \varphi(a_0, \ldots, a_n)\}$ is computable.

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A model $\mathfrak{A}$ ***has computable presentation*** (is said to be ***computably presentable***) if it is isomorphic to a computable model.

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Denote by $\mathfrak{M}_p$ the set of all (isomorphic) prime models over realizations of p, i. e.

$$\mathfrak{M}_p = \{\mathfrak{M}_{\bar{a}} \mid \langle \mathfrak{M}_{\bar{a}}, \bar{a} \rangle \text{ is a prime model of } Th(\mathfrak{M}, \bar{a}), \text{ where } \mathfrak{M} \models p(\bar{a})\}.$$

Definition

A type p ***does not exceed*** a type q ***under the Rudin-Keisler pre-order*** (p ***is dominated by*** q) if $\mathfrak{M}_q \models p$. Written $p \leqslant_{RK} q$.
$p \sim_{RK} q \Leftrightarrow (p \leqslant_{RK} q \,\&\, q \leqslant_{RK} p)$.
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Denote by $RK(T)$ the set of all types of isomorphism of $\mathfrak{M}_p$, throughout all $p \in S(T)$.
This set is pre-ordered by the relation $\leqslant_{RK}$.

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A model $\mathfrak{M}$ is said to be ***limit over a type*** p if $\mathfrak{M} = \bigcup_{n\in\omega} \mathfrak{M}_n$, for some elementary chain $(\mathfrak{M}_n)_{n\in\omega}$ over p, and $\mathfrak{M} \ncong \mathfrak{M}_p$.

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Consider $\widetilde{\mathbf{M}} \in RK(T)/\sim_{RK}$. Let $\widetilde{\mathbf{M}} = \{\mathfrak{M}_{p_0}, \ldots, \mathfrak{M}_{p_n}\}$. Denote by $IL(\widetilde{\mathbf{M}})$ the number of two by two non-isomorphic models each of which is limit over some type p_i.

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Theorem (S. Sudoplatov)

The following conditions are equivalent:

1. $\omega(T) < \omega$;
2. $|S(T)| = \omega$, $|RK(T)| < \omega$, $IL(\widetilde{\mathbf{M}}) < \omega$, *for any* $\widetilde{\mathbf{M}} \in RK(T)/\sim_{RK}$.

Definition

Let $\langle X; \leqslant \rangle$ is finite pre-ordered set with the least element x_0 and the greatest class $\widetilde{x}_n$ in ordered factor-set $\langle X; \leqslant \rangle / \sim$ (where $x \sim y \Leftrightarrow x \leqslant y$ and $y \leqslant x$), $\widetilde{x}_0 \neq \widetilde{x}_n$. Let $f : X/\sim \rightarrow \omega$ is a function, satisfying next properties $f(\widetilde{x}_0) = 0$, $f(\widetilde{x}_n) > 0$, $f(\widetilde{y}) > 0$, when $|\widetilde{y}| > 1$. The pair (X, f) is said to be ***e-parameters***. At that, the set X is said to be ***the first e-parameter*** and the function f — ***the second e-parameter***.

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Definition

A theory T is said to be ***Ehrenfeucht theory with e-parameters*** (X, f) if there exists an isomorphism $\varphi : X \to RK(T)$ and for any $\widetilde{x} \in X/\sim$, an equality $IL(\varphi(\widetilde{x})) = f(\widetilde{x})$ holds.

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Definition

Spectrum of decidable models of Ehrenfeucht theory $\mathbb{SDM}(T)$ is a pair (Y, g), where $Y = \{x \in X \mid$ element x corresponds to a decidable model of the theory $T\}$ (corresponds — in terms of isomorphism φ form previous definition); $\delta f = \delta g$ (δ is domain of a function), ($g(x) = m \Leftrightarrow$ there exist exactly m decidable limit non-isomorphic models of T over the model, corresponding to x).

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Spectrum of computable models of Ehrenfeucht theory $\mathbb{SCM}(T)$ is a pair (Y, g), where $Y = \{x \in X \mid$ element x corresponds to a computable model of the theory $T\}$; $\delta f = \delta g$, ($g(x) = m \Leftrightarrow$ there exist exactly m computable limit non-isomorphic models of T over the model, corresponding to x).

Problem

Describe sets $\mathbb{SDM}(T)$ *and* $\mathbb{SCM}(T)$ *for arbitrary Ehrenfeucht theory* T.

Denote by L_n a linear ordered set, composed of $n+1$ elements: $\{x_0 < x_1 < \ldots < x_n\}$.

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Theorem

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Let $1 \leqslant n \in \omega$, $0 \leqslant k \leqslant n$. There exists Ehrenfeucht theory T_n for which $RK(T_n) \cong L_n$ holds. At that, models, corresponding to elements $x_0, x_1, \ldots, x_k$ from L_n, are decidable, models, corresponding to elements $x_{k+1}, \ldots, x_n$, have no computable presentations.

Theorem

For all $1 \leqslant m \in \omega$, *there exists Ehrenfeucht theory* T_m, *such that* $RK(T_m) \cong L_m$, *every quasi-prime model of* T_m *is not computably presentable and there exists computably presentable model of* T_m.

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Corollary

For all $1 \leqslant m \in \omega$, *there exists Ehrenfeucht theory* T_m, $RK(T_m) \cong L_m$, *such that a model* $\mathfrak{M} \models T_m$ *have computable presentation if and only if* $\mathfrak{M}$ *is limit model over powerful type of the theory* T_m.

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For all $1 \leqslant m \in \omega$, *there exists Ehrenfeucht theory* T_m, $RK(T_m) \cong L_m$, *such that every quasi-prime model of* T_m *have no computable presentation, every limit model of* T_m, *have computable presentation.*

Thank you
for attention!