Survey of Degree Spectra of High$_n$ and Non-low$_n$ Degrees

Russell Miller,

Queens College & Graduate Center

C.U.N.Y.

July 31, 2009

Logic Colloquium 2009
Sofia University

Spectrum of a Structure

Defns: For a countable structure $\mathcal{S}$ with domain ω, the *Turing degree of* $\mathcal{S}$ is the Turing degree of the atomic diagram of $\mathcal{S}$. The *spectrum of* $\mathcal{S}$ is

$$\mathrm{Spec}(\mathcal{S}) = \{\deg(\mathcal{A}) : \mathcal{A} \cong \mathcal{S}\}$$

of all Turing degrees of copies of $\mathcal{S}$.

For a relation R on a computable structure $\mathcal{M}$, the *spectrum of* R, $\mathrm{DgSp}_{\mathcal{M}}(R)$, is

$$\{\deg(f(R)) : f : \mathcal{M} \cong \mathcal{N} \ \&\ \mathcal{N} \text{ is computable}\}.$$

Non-low Degrees

Theorem (Slaman; Wehner; Hirschfeldt): There exists a structure whose spectrum contains every Turing degree $> \mathbf{0}$, but not the degree $\mathbf{0}$.

This also holds with an arbitrary $\boldsymbol{d}$ in place of $\mathbf{0}$.

Theorem (GHKMMS): For each $n \in \omega$, there exists a structure whose spectrum contains precisely the non-low$_n$ degrees. Indeed, for each computable successor ordinal α, there exists a structure with spectrum

$$\{\deg(X) : (\exists \mathbf{d} \notin \mathbf{\Delta}^{\mathbf{0}}_{\alpha})[\mathbf{d} \text{ is } \Delta^0_\alpha \text{ relative to } \mathbf{X}]\}.$$

High Degrees

Given a structure $\mathcal{A}$, the technique of (GHKMMS) builds, for each successor ordinal α, a structure $\mathcal{B}$ such that

$$\boldsymbol{c} \in \operatorname{Spec}(\mathcal{B}) \iff \boldsymbol{c}^{(\alpha)} \in \operatorname{Spec}(\mathcal{A}).$$

Fact: For every $\boldsymbol{d}$, there is a structure $\mathcal{A}_{\boldsymbol{d}}$ with spectrum $\{\boldsymbol{c} : \boldsymbol{c} \geq_T \boldsymbol{d}\}$.

So with $\alpha = 1$ and $\boldsymbol{d} = \mathbf{0}''$, this gives a structure $\mathcal{B}$ whose spectrum contains exactly the high-or-above degrees (those $\boldsymbol{c}$ with $\boldsymbol{c}' \geq_T \mathbf{0}''$). Likewise for high$_n$, and even high$_\alpha$ (with $\alpha \notin \mathbf{LOR}$). This extends a known result...

Spectrum of high degrees

Proposition (Harizanov-Miller): There exists a relation R on a computable dense linear order $\mathbb{Q}$ with

$$\mathrm{DgSp}_{\mathbb{Q}}(R) = \{\mathbf{d} : \mathbf{d}' \geq_{\mathbf{T}} \mathbf{0}''\}.$$

Corollary: There exists a structure with this same spectrum.

Corollary: Not all spectra of unary relations on $(\mathbb{Q}, <)$ can be realized as spectra of linear orders.
Proof: By a theorem of Knight (1986), $\mathbf{0}'$ is the only possible jump degree of a linear order.

How About Linear Orders?

For Boolean algebras, the spectrum $\{\boldsymbol{d} : \boldsymbol{d} > \mathbf{0}\}$ is impossible.

- Jockusch-Soare: For every c.e. $\boldsymbol{d} > \mathbf{0}$, there exists a linear order whose spectrum contains $\boldsymbol{d}$ but not $\mathbf{0}$.
- Downey/Seetapun: Extension to any $\boldsymbol{d}$ with $\mathbf{0} < \boldsymbol{d} \leq \mathbf{0}'$.
- Knight: Extension to any $\boldsymbol{d} > \mathbf{0}$.
- M.: There is a single linear order whose spectrum contains all $\boldsymbol{d}$ with $\mathbf{0} < \boldsymbol{d} \leq \mathbf{0}'$, but not $\mathbf{0}$.
- Frolov: For each $n \in \omega$, there is a linear order whose spectrum contains all non-low$_n$ degrees $\leq \mathbf{0}'$ but no low$_n$ degrees.

Question: Can a linear order have a spectrum of precisely the non-low$_n$ degrees?

Where Next?

Frolov's result builds an order $\mathcal{L}$ by relativizing Miller's result to $\mathbf{0}^{(n)}$, so that $\mathrm{Spec}(\mathcal{L})$ contains all $\boldsymbol{d}$ with $\mathbf{0}^{(n)} < \boldsymbol{d} \leq \mathbf{0}^{(n+1)}$, but not $\mathbf{0}^{(n)}$.

Then it applies the relativization of a theorem of Downey and Knight: A linear order $\mathcal{L}$ is Δ_2^0 iff $(\eta + 2 + \eta) \cdot \mathcal{L}$ is computable.

So the order $\mathcal{L}_n = (\eta + 2 + \eta)^n \cdot \mathcal{L}$ has all non-low$_n$ degrees below $\mathbf{0}'$ in its spectrum, but no low$_n$ degree.

Spectral Universality

An embedding $f : \mathcal{A} \hookrightarrow \mathcal{B}$ *preserves the spectrum* if $\mathrm{Spec}(A) = \mathrm{DgSp}_{\mathcal{B}}(f(\mathcal{A}))$.

A computable model $\mathcal{B}$ of a theory T is *spectrally universal* for T if every countable model $\mathcal{A}$ of T embeds into $\mathcal{B}$ via some f preserving the spectrum.

Many (but not all!) computable Fraïssé limits of theories turn out to be spectrally universal for those theories. Examples:

- Countable dense linear order.
- Random graph.
- Countable atomless Boolean algebra.

Counterexample:

- Algebraic closure of the field $\mathbb{Z}/(p)$.

Structures vs. Relations

Corollary (of the spectral universality of the random graph): The spectra of countable graphs are precisely the spectra of unary relations on the random graph.

We saw above that this does *not* hold of linear orders. Spectral universality of the countable DLO shows that every spectrum of a LO is the spectrum of a unary relation on the DLO, but the converse is false.

New Possible Counterexample

Theorem (M.): There exists a unary relation $\tilde{R}$ on the countable DLO $(\mathbb{Q}, \prec)$ whose degree spectrum contains the non-low degrees:

$$\mathrm{DgSp}_{\mathbb{Q}}(\tilde{R}) = \{\mathbf{d} : \mathbf{d}' >_{\mathbf{T}} \mathbf{0}'\}.$$

For a given finite set $F = \{n_1, n_2, \ldots, n_k\} \subset \omega$ and $a \prec b$ in $\mathbb{Q}$, define $\tilde{R}$ on (a, b) for F as follows:

Wehner's construction gives a family $\mathcal{F}$ of finite sets F, uniformly enumerable by any degree $> \mathbf{0}'$, but not by $\mathbf{0}'$.
For each $F \in \mathcal{F}$, in each order, define $\tilde{R}$ as above for this F on densely many intervals (a, b) in $\mathbb{Q}$.

Does this work?

This $\tilde{R}$ on $(\mathbb{Q}, \prec)$ gives a potential second counterexample to the converse of spectral universality for linear orders.

Question: Does there exist a linear order with spectrum $\{\boldsymbol{c} : \boldsymbol{c}' >_T \mathbf{0}'\}$?

Notice that the restriction of $\prec$ to $\tilde{R}$ does *not* yield such an order.

Fields

Example: Fix $r_0 = e$ and $r_{i+1} = e^{r_i}$. Given $S \subseteq \omega$, let F_S be the closure of $\mathbb{Q}(r_t \mid t \in S)$ under square roots of positive elements. We claim that

$$\mathrm{Spec}(F_S) = \{\boldsymbol{d} : S \text{ is } \Sigma^0_2 \text{ in } \boldsymbol{d}\}.$$

Cor.: For any $A \subseteq \omega$, there is a field whose spectrum contains precisely those $\boldsymbol{d}$ with $A \leq \boldsymbol{d}'$.

$$\begin{aligned}\mathrm{Spec}(F_{A'}) &= \{\boldsymbol{d} : (\exists D \in \boldsymbol{d}) A' \leq_1 D''\} \\ &= \{\boldsymbol{d} : (\exists D \in \boldsymbol{d}) A \leq_T D'\}\end{aligned}$$

As a relation on its algebraic closure, F_S has the same spectrum $\{\boldsymbol{d} : S \text{ is } \Sigma^0_2 \text{ in } \boldsymbol{d}\}$.

So the high degrees form the spectrum of a field, and also the spectrum of a subfield of the algebraic closure.

$\mathbf{Spec}(F_S) = \{\boldsymbol{d} : S$ is Σ^0_2 in $\boldsymbol{d}\}$.

$\subseteq$: If $E \cong F_S$, then S is the set

$$\{t \in \omega : (\exists x \in E)(\forall q \in \mathbb{Q})[q < r_t \leftrightarrow q \prec x \text{ in } E]\}.$$

The order $\prec$ on E is E-computable, by the closure of E under square roots of positive elements.

$\supseteq$: If $S \leq_1 \mathrm{Fin}^D$, let $t \in S$ iff $W^D_{h(t)}$ is finite. Start building F_ω (the field containing all r_t). Each time $W^D_{h(t)}$ receives an element, make the old r_t become rational and add a new r_t to replace it.