On forcing with σ-ideals of closed sets

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Idealized forcing

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Another way

Note that the forcing $\mathbf{P}_I = \mathrm{Bor}(X) \setminus I$ is equivalent to the quotient Boolean algebra $\mathrm{Bor}(X)/I$ (which is the separative quotient of $\mathbf{P}_I$).

The generic real

A forcing notion of the form $\mathrm{Bor}(\omega^\omega)/I$ adds the *generic real*, denoted $\dot{g}$ and defined in the following way:

$$[\![\dot{g}(n) = m]\!] = [(n,m)]_I$$

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Genericity

Of course, the generic ultrafilter can be recovered from the generic real in the following way:

$$G = \{B \in \mathrm{Bor}(X) : g \in B\}$$

where g denotes the generic real.

The σ-ideal

We say that a σ-ideal I is *generated by closed sets*, if for each $A \in I$ there is a sequence of closed sets $F_n \in I$ such that $A \subseteq \bigcup_{n<\omega} F_n$.

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Theorem (Solecki)

Let I be a σ-ideal generated by closed sets. If $A \subseteq X$ is analytic, then either $A \in I$, or else A contains a G_δ set G such that $G \notin I$.

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Corollary

From the above theorem of Solecki we get that if I is generated by closed sets, then $\mathbf{P}_I$ is forcing equivalent to $\mathbf{Q}_I = \mathbf{\Sigma}^1_1 \setminus I$ ($\mathbf{P}_I$ is dense in $\mathbf{Q}_I$).

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Axiom A

Recall that a forcing notion $\mathbf{P}$ satisties Baumgartner's *Axiom A* if there is a sequence of partial orders $\leq_n$ on $\mathbf{P}$ such that $\leq_0 = \leq$, $\leq_{n+1} \subseteq \leq_n$ and

- if $\langle p_n \in \mathbf{P}, n < \omega \rangle$ is such that $p_{n+1} \leq_n p_n$, then there is $q \in \mathbf{P}$ such that $q \leq_n p_n$ for all n,
- for every $p \in \mathbf{P}$, for every n and for every name $\dot{\alpha}$ for an ordinal there exist $q \in \mathbf{P}$ and a countable set of ordinals A such that $q \leq_n p_n$ for each $n < \omega$, and $q \Vdash \dot{\alpha} \in A$.

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Sketch of the proof

Assume $X = \omega^\omega$ and fix I. Let $A \subseteq \omega^\omega$ be an analytic set and let T be a tree on $\omega \times \omega$ projecting to A.

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Game $G_I(T)$

Consider the following game (between Adam and Eve).

- in his n-th move, Adam picks $\tau_n \in T$ such that τ_{n+1} extends τ_n.
- in her n-th move, Eve picks a clopen set O_n in ω^ω such that

$$\mathrm{proj}[T_{\tau_n}] \notin \mathcal{I} \ \Rightarrow \ O_n \cap \mathrm{proj}[T_{\tau_n}] \notin \mathcal{I}.$$

Winning condition

By the end of a play, Adam and Eve have a sequence of closed sets E_k in ω^ω defined as follows:

$$E_k = 2^\omega \setminus \bigcup_{i<\omega} O_{\rho^{-1}(i,k)}.$$

(ρ is some fixed bijection between ω and ω^2). Define $x = \pi(\bigcup_{n<\omega} \tau_n) \in \omega^\omega$. **Adam wins** if and only if

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Lemma

Eve has a winning strategy in $G_I(T)$ if and only if $A = \mathrm{proj}[T] \in I$.

Strategy

If S is a strategy for Adam in $G_I(T)$, then by $\mathrm{proj}[S]$ we denote the set of points $x \in \omega^\omega$ which arise at the end of some game obeying S.

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Forcing with strategies

Consider the following forcing $\mathbf{T}_I$:
$\{S : S$ is a winning strategy for Adam in $G_I(T)$ for some tree $T\}$
ordered as follows:

$$S_0 \leq S_1 \text{ iff } \mathrm{proj}[S_0] \subseteq \mathrm{proj}[S_1].$$

Dense embedding

Notice that $\mathbf{T}_I \ni S \mapsto \mathrm{proj}[S] \in \mathbf{Q}_I$ is a dense embedding, hence the three forcing notions $\mathbf{P}_I$, $\mathbf{Q}_I$ and $\mathbf{T}_I$ are forcing equivalent. Let us show that $\mathbf{T}_I$ satisfies Axiom A.

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Winning condition revised

Recall that the winning condition for Adam in $G_I(T)$ says that

$$x \notin \bigcup_k E_k.$$

Fix k. For each play in $G_I(T)$ both x and E_k are built "step-by-step" (E_k from basic clopen sets which sum up to $\omega^\omega \setminus E_k$). Hence, if π is a play and $x \notin E_k$, then there is $m < \omega$ such that the partial play $\pi \restriction m$ already determines that "$x \notin E_k$".

Fusion

Let $S \in \mathbf{T}_I$ be a winning strategy for Adam. For each play π in S there is the least $m < \omega$ such that $\pi \restriction m$ determines that "$x \notin E_i$" for $i \leq k$. Therefore, we can define the k-th front of the tree S, denoted by $F_k(S)$ so that each play determines "$x \notin E_i$" before passing through $F_k(S)$.

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Axiom A

We define the inequalities $\leq_k$ as follows: $S_1 \leq_k S_0$ if and only if

- $S_1 \leq S_0$,
- $F_k(S_1) = F_k(S_0)$.

The end

Thank You.