Universal Fragments of some Region-based Theories of Space

Tinko Tinchev

Faculty of Mathematics and Informatics
Sofia University

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Let $\mathcal{T} = \langle T, \tau \rangle$ be a topological space, *Cl* and *Int* be its closure and interior operators.
A subset X of $\mathcal{T}$ is called *regular closed* iff $X = Cl(Int(X))$; X is called *regular open* iff $X = Int(Cl(X))$.
Let $0 = \varnothing$, $1 = T$ and $X_1 \leq X_2$ iff $X_1 \subseteq X_2$. Then the regular closed (open) sets forms a Boolean algebra under $\leq$ with top element 1 and bottom element 0: $RC(\mathcal{T})$ (resp. $RO(\mathcal{T})$).

Remark that in $RC(\mathcal{T})$ the meet $\sqcup$ coincide with set-theoretical union $\cup$, but the join $X_1 \sqcap X_2$ and complement X^* are $Cl(Int(X_1 \cap X_2))$ and $Cl(T \setminus X)$, respectively.

regions [2]

The regions: the elements of the Boolean algebra $RC(\mathcal{T})$.
Special case: $\mathcal{T} = \mathbb{R}^m$, i.e. the regions form the Boolean algebra $RC(\mathbb{R}^m)$, $m \geq 1$.
The Boolean algebra of the *polytops* in $\mathbb{R}^m$, $PRC(\mathbb{R}^m)$: the subalgebra of $RC(\mathbb{R}^m)$ generated by the set of *basic polytops*, where basic polytop is finite join of closed half-spaces of $\mathbb{R}^m$.
In other words, the polytop is finite join of finite union of basic polytops.

k-contact

Let $a_1, \ldots, a_k$, $k \geq 2$, be regions. They are in *k-contact*, $\mathbf{C}_k(a_1, \ldots, a_k)$ iff $a_1 \cap \cdots \cap a_k \neq \varnothing$.

$\mathfrak{A}_B = (B, C^2, C^3, \ldots)$, where B is a Boolean algebra of regions

The first order language $\mathcal{L}$ is the extension of the language of the Boolean algebras, $0, 1, \sqcup, \sqcap, {}^*$ with the set of k-ary predicate symbols C^k for all $k > 1$.

Let $\mathcal{K}$ be a class of Boolean algebras of regions and

$$Th_{\forall}(\mathcal{K}) = \{\phi \mid \mathfrak{A}_B \models \phi, B \in \mathcal{K}, \phi \text{ is sentence}\}$$

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Our aim is to axiomatize:

1. $Th_{\forall}(\mathcal{K}_{all})$, where $\mathcal{K}_{all}$ is the class of all $RC(\mathcal{T})$

2. $Th_{\forall}(\mathcal{K}_{connected})$, where $\mathcal{K}_{connected}$ is the class of all $RC(\mathcal{T})$ for connected topological spaces $\mathcal{T}$

3. $Th_{\forall}(RC(\mathbb{R}^m))$, $m = 1$, $m > 1$

4. $Th_{\forall}(PRC(\mathbb{R}^m))$, $m = 1$, $m > 1$

and to give a new proof of:

5. $Th_{\forall}(\mathcal{K}_{connected}) = Th_{\forall}(RC(\mathbb{R}^2)) = Th_{\forall}(PRC(\mathbb{R}^2)) = Th_{\forall}(RC(\mathbb{R}^m)) = Th_{\forall}(PRC(\mathbb{R}^m))$, $m \geq 2$

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Let T_{all} be

- the set an universal axiomatization of the Boolean algebras
- the axioms for the equality in $\mathcal{L}$ +
- universal closure of the following formulas

$C^k(x_1, \dots, x_k) \rightarrow \bigwedge_{i=1}^{k}(x_i \neq 0)$

$C^k(x_1, \dots, x' \sqcup x'', \dots, x_k) \leftrightarrow$
$\bigwedge C^k(x_1, \dots, x', \dots, x_k) \vee C^k(x_1, \dots, x'', \dots, x_k)$, $1 \leq i \leq k$

$(x \neq 0) \rightarrow C^k(x, \dots, x)$ (sufficient $k = 2$)

$C^k(x_1, \dots, x_k) \rightarrow C^k(x_{\sigma(1)}, \dots, x_{\sigma(k)})$, where σ is a permutation of $1, \dots, k$

$C^k(x_1, \dots, x_k) \rightarrow C^{k+1}(x_1, \dots, x_k, x_k)$

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Theorem

Let ϕ be an universal sentence from $\mathcal{L}$. Then

$$T_{all} \vdash \phi \iff \phi \in Th_{\forall}(\mathcal{K}_{all})$$

Let $T_{connected}$ be $T_{all} + \forall x((x \neq 0) \wedge (x \neq 1) \rightarrow C^2(x, x^*))$

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Let ϕ be an universal sentence from $\mathcal{L}$. Then

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Idea: I

1) To consider open formulae as modal formulae; Kripke frame $(W, R^2, R^3, ...)$ with natural conditions for relations R^k

2) To define the analog of Boolean Contact Algebras — eBCA

3) Finite eBCA's are isomorphic with the Boolean algebras of subsets and the relations

$C^k_{R^k}(X_1, \ldots, X_k)$ iff $(\exists x_1 \in X_1)$,... $(\exists x_k \in X_k)$ s.t. $R^k(x_1, \ldots, x_k)$

Lemma

The minimal modal logic L_{min} is complete with respect to the class of all finite frames.

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The logic $L_{min} + (a \neq 0) \wedge (a^* \neq 0) \to C^2(a, a^*)$ is complete with respect to the class of all finite connected with respect to R^2 Kripke frames.

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Idea: II

Lemma

For any finite eBCA **B** *there is a regular closed set* $X \subseteq \mathbb{R}^3$ *such that* **B** *is isomorphic with the subalgebra of* $RC(\mathbf{X})$ *where* **X** *is the set* X *with induces topology.*

We “realize” the frame corresponding to **B** with regions in $\mathbb{R}^3$.

The Kripke frame corresponding to **B** is connected with respect to R^2 iff **X** is connected (iff X is connected in $\mathbb{R}^3$).

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Let T_1 be $T_{connected} +$ an universal closure of
$C^3(x_1, x_2, x_3) \to \bigvee_{1 \leq i < j \leq j}(x_i \sqcap x_j \neq 0)$ and
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Theorem

Let ϕ be an universal sentence from $\mathcal{L}$. Then

$$T_1 \vdash \phi \iff \phi \in Th_\forall(PRC(\mathbb{R}))$$

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1) The corresponding modal logic is complete with respect to the class of all finite connected frames satisfying a trivial condition for R^k, $k \geq 3$.

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Idea:

1) The corresponding modal logic is complete with respect to the class of all finite connected frames.

2) Use "appropriate" p-morphic preimage: finite tree with respect to R^2 and special kind conditions for R^k, $k > 2$

3) Realize this finite tree as finite partition of $\mathbb{R}^2$ using closed bands.

4) Realize this finite tree as, generally infinite, partition of $\mathbb{R}$.

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Thank you very much!