Splitting properties in 2-c.e. degrees.

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Definitions and conventions.

All sets are subsets of the set of natural numbers $\omega = \{0, 1, 2...\}$. If a set $A \subseteq \omega$ is Turing reducible to $B \subseteq \omega$ then we denote $A \leq_T B$.

$A \equiv_T B$ iff $A \leq_T B$ and $B \leq_T A$.

$\mathbf{a} = deg(A) = \{B \mid B \equiv_T A\}$.

The degrees with "$\leq$" and "$\cup$" form an upper semilattice, where $\mathbf{a} \cup \mathbf{b} = deg(A \oplus B)$ and $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.

Also in this structure a jump operator is defined such that $\mathbf{b} \leq \mathbf{a} \rightarrow \mathbf{b}' \leq \mathbf{a}'$.

We will consider only Turing degrees $\leq \mathbf{0}'$, where $\mathbf{0}'$=deg(K) is the degree of halting problem.

Let a set $A \leq_T K$, so $A(x) = lim_s f(x,s)$, $f(x,0) = 0$, where f is a computable function. A set A is n-computable enumerable (c.e.), if for any x $|\{s|f(x,s) \neq f(x,s+1)\}| \leq n$. The degree of the set $\mathbf{a} = deg(A)$ is n-c.e.; if it also doesn't consist $(n-1)$-c.e. sets, then is has a properly n-c.e. degree.

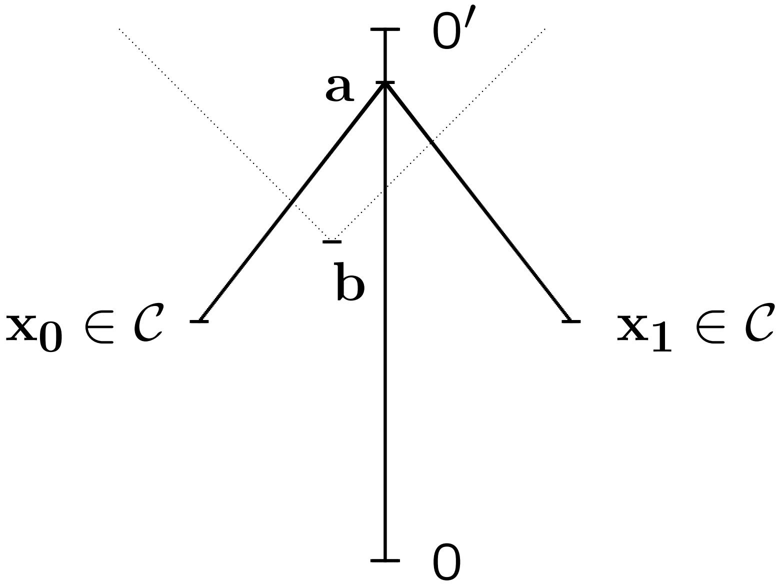
Definition. Degree $\mathbf{a}$ is splittable in a class of degrees $\mathcal{C}$ if there exist degrees $\mathbf{x_0}, \mathbf{x_1} \in \mathcal{C}$ such that $\mathbf{a} = \mathbf{x_0} \cup \mathbf{x_1}$ and $\mathbf{x_0}, \mathbf{x_1} < \mathbf{a}$.

Definition. For a given degrees $\mathbf{x}$ and $\mathbf{y}$ we say that that the degree $\mathbf{x}$ avoids the upper (lower) cone of $\mathbf{y}$ if $\mathbf{y} \not\leq \mathbf{x}$ ($\mathbf{x} \not\leq \mathbf{y}$).

Given degrees $\mathbf{0} < \mathbf{b} < \mathbf{a}$ and a splitting of $\mathbf{a} = \mathbf{x_0} \cup \mathbf{x_1}$

Definition. If $\mathbf{b} \not\leq \mathbf{x_i} (i = 0, 1)$ then $\mathbf{a}$ is splittable avoiding upper cone of $\mathbf{b}$.

Definition. If $\mathbf{b} \leq \mathbf{x_i} (i = 0, 1)$ then $\mathbf{a}$ is splittable above $\mathbf{b}$.



By default we assume that $\mathcal{C}$ is the smallest class containing $\mathbf{a}$. E.g., in the finite levels of Ershov's hierarchy we usually try to split in the same level.

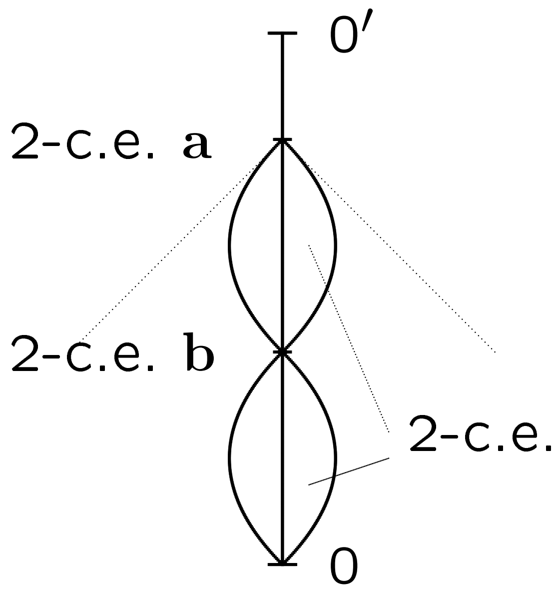
[Sacks, 1963] Splitting of c.e. degrees (can be generalized to avoid upper cone of any noncomputable Δ^0_2-degree).

[Robinson, $\approx$ 1970] Splitting of c.e. degrees above low c.e. degrees.

[Arslanov, Cooper, Li; 1992, 2002, 2004] Splitting of 2-c.e. degrees. Splitting above c.e. degrees, splitting above low 2-c.e. degrees.

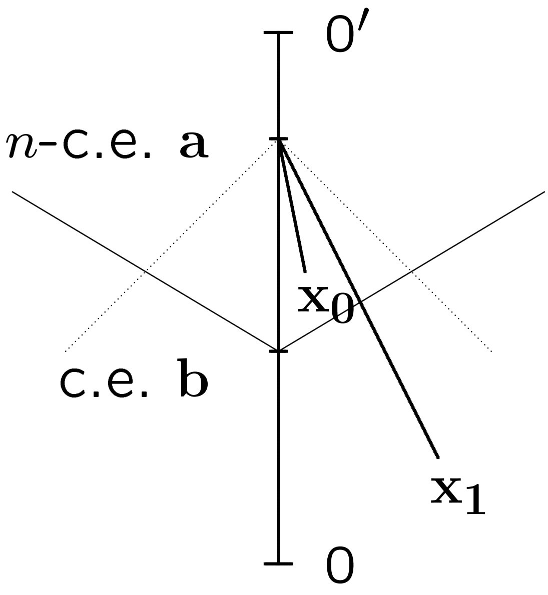
Another direction of research is splitting with avoiding cones. Theorem 1 provides sufficient conditions for a properly 2-c.e. degree $\mathbf{a}$ to be splitted avoiding upper cone of Δ^0_2 degree $\mathbf{d}$. In general case it's not possible since to the theorem of Arslanov, Kalimullin and Lempp (also it follows from the theorem of Cooper and Li or Thereom 3 provided below).

[Arslanov, Kalimullin, Lempp, 2003] There exist noncomputable 2-c.e. degrees $\mathbf{b} < \mathbf{a}$ such that for any 2-c.e. degree $\mathbf{v}$: $\mathbf{v} \leq \mathbf{a} \longrightarrow ([\mathbf{v} \leq \mathbf{b}] \vee [\mathbf{b} \leq \mathbf{v}])$.



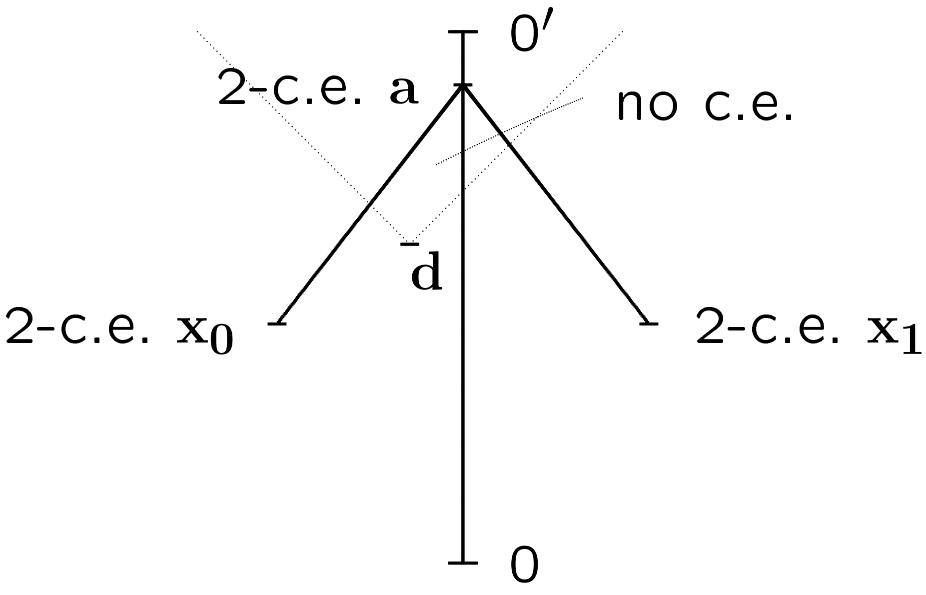
It is known as "bubble". Notice, that the middle degree $\mathbf{b}$ is c.e. degree.

[Cooper, Li, 2004] For any $n \geq 2$ there exist n-c.e. degree $\mathbf{a}$, c.e. degree $\mathbf{b}$ such that $\mathbf{0} < \mathbf{b} < \mathbf{a}$ and such that for any n-c.e. degrees $\mathbf{x_0}$ and $\mathbf{x_1}$: $\mathbf{a} = \mathbf{x_0} \cup \mathbf{x_1} \longrightarrow ([\mathbf{b} \leq \mathbf{x_0}] \vee [\mathbf{b} \leq \mathbf{x_1}])$.



Sufficient conditions for a 2-c.e. degree $\mathbf{a}$ to be splittable avoiding upper cone of Δ_2^0 degree below it.

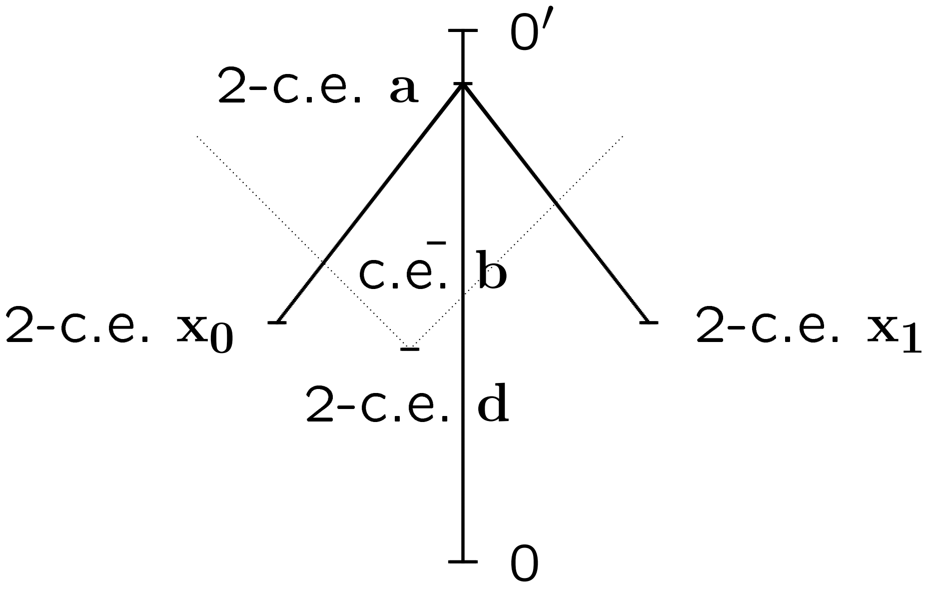
Theorem 1. *Let* $\mathbf{a}$ *and* $\mathbf{d}$ *be properly 2-c.e. degrees such that* $\mathbf{0} < \mathbf{d} < \mathbf{a}$ *and there are no c.e. degrees between* $\mathbf{a}$ *and* $\mathbf{d}$*. Then* $\mathbf{a}$ *is splittable avoiding upper cone of* $\mathbf{d}$*.*



Theorem 1 generalizes Cooper's splitting theorem in 2-c.e. degrees. Also it generalizes Sacks's splitting theorem in c.e. degrees in the following sense: we can consider 2-c.e. degrees instead of c.e. and c.e. degree instead of computable degree (we will have the same type of isolating).

The question arises about a characterization, which could express the isolation in terms of splitting and vice versa. One may assume that if a 2-c.e. degree $\mathbf{a}$ above $\mathbf{d}$ is splittable avoiding the upper cone of $\mathbf{d}$ then there are no c.e. degrees between $\mathbf{d}$ and $\mathbf{a}$. The above mentioned "the bubble existence theorem" can be considered as a confirmation of this assumption. But Theorem 2 shows that this doesn't hold.

Theorem 2. *There exist a c.e. degree* $\mathbf{b}$, *2-c.e. degrees* $\mathbf{d}$, $\mathbf{a}$, $\mathbf{x_0}$, $\mathbf{x_1}$ *such that* $\mathbf{0} < \mathbf{d} < \mathbf{b} < \mathbf{a}$, $\mathbf{a} = \mathbf{x_0} \cup \mathbf{x_1}$, $\mathbf{x_0} < \mathbf{a}$, $\mathbf{x_1} < \mathbf{a}$, $\mathbf{d} \not\leq \mathbf{x_0}$, $\mathbf{d} \not\leq \mathbf{x_1}$ *and* $\mathbf{d}$ *and* $\mathbf{a}$ *have properly 2-c.e. degrees.*



Sketch of the proof of Theorem 2.

Note that considering a c.e. degree $\mathbf{c}$ instead of the degree $\mathbf{d}$ we can construct sets A, B, C, X_0, X_1 and assign corresponding degrees $\mathbf{c}$=deg(C), $\mathbf{b}$=deg($C \oplus B$), $\mathbf{a}$=deg($C \oplus B \oplus A$), $\mathbf{x_0}$=deg(X_0), $\mathbf{x_1}$=deg(X_1). Then it follows from the weak density theorem (Cooper, Lempp, Watson, 1989]) that there exists a properly 2-c.e. degree $\mathbf{d}$ such that $\mathbf{c} < \mathbf{d} < \mathbf{b}$. The degree $\mathbf{d}$ is the desired degree.

Therefore, it's enough to construct sets A, B, C, X_0, X_1, satisfying the following requirements (we construct sets X_0, X_1 avoiding the lower cone of C for uniformity).

$\mathcal{R}_e$: $X_0 \oplus X_1 \not\equiv_T W_e$;

$\mathcal{S}^C_{2e}$: $X_0 \neq \Phi^C_e$;

$\mathcal{S}^C_{2e+1}$: $X_1 \neq \Phi^C_e$;

$\mathcal{S}^X_{2e}$: $C \neq \Phi^{X_0}_e$;

$\mathcal{S}^X_{2e+1}$: $C \neq \Phi^{X_1}_e$;

$\mathcal{N}_e$: $B \neq \Phi^C_e$;

$\mathcal{T}$: $B \oplus C \leq_T X_0 \oplus X_1$.

For the requirement $\mathcal{T}$ we define
$A = X_0 \oplus X_1$ and
$\deg(C \oplus B \oplus A) = \deg(A)$.

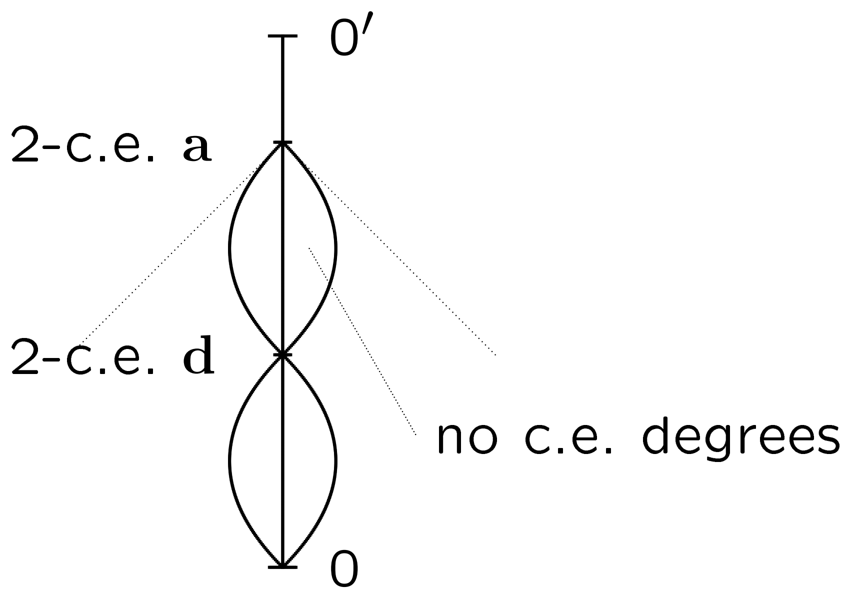
The strategy for the requirement $\mathcal{S}^X_{2e}$ takes in attention the requirement $\mathcal{T}$. Assigning a witness y we define a computable function-marker $\alpha(y)$, and enumerating y into C we enumerate the marker $\alpha(y)$ into X_1. The same for requirements $\mathcal{N}_e$.

Corollaries of Theorem 1.

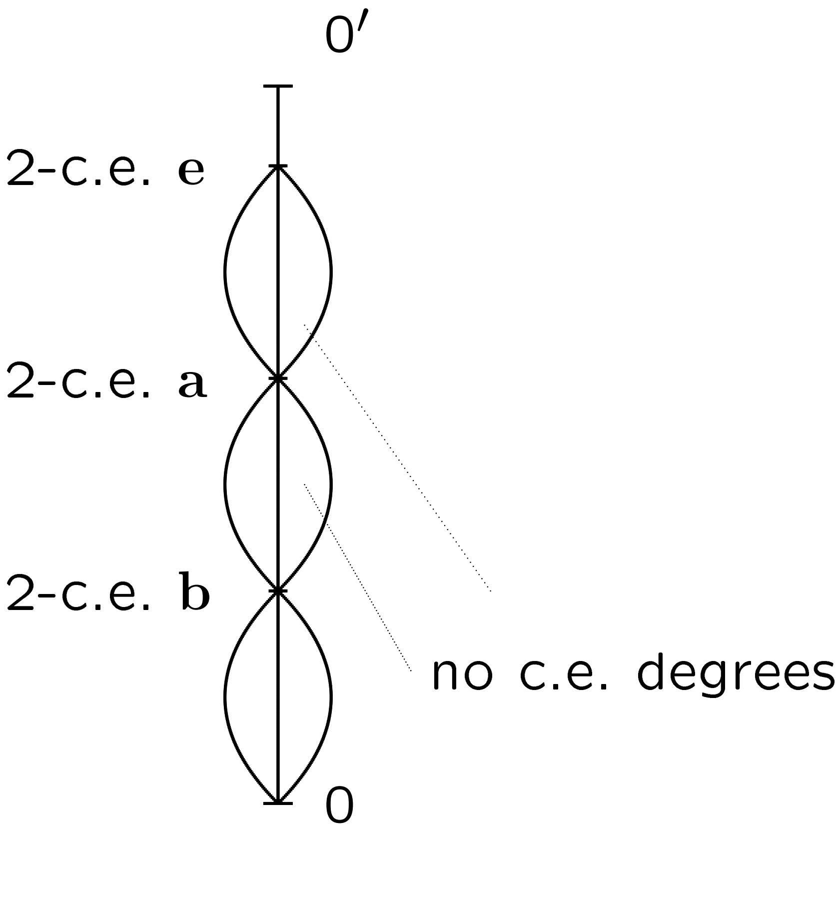
Middle of the "bubble" is c.e. degree.
Proof.

1) There no c.e. degrees between $\mathbf{d}$ and $\mathbf{b}$, otherwise we can split it by Sacks's splitting theorem.

2) If $\mathbf{d}$ has properly 2-c.e. degree then we apply theorem 1 and the previous statement 1. So, contradiction again.



There are no "3-bubbles" in 2-c.e. degrees. Because of previous corollary the degrees $\mathbf{a}$ and $\mathbf{b}$ are c.e. So, we can apply to $\mathbf{a}$ Sacks's splitting theorem.

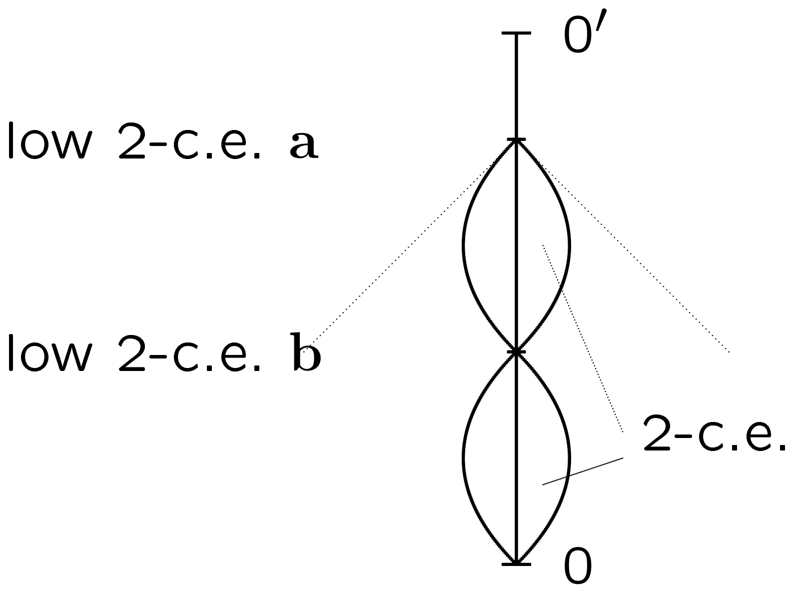


Definition.

A set A is low if $A' \equiv_T K$. A set A is n-low for $n > 1$ if $A^{(n)} \equiv_T K^{(n-1)}$. Respectively degrees $\mathbf{a} = deg(A)$ are low (n-low).

The following theorem shows that "bubble" could be constructed in low 2-c.e. degrees.

Theorem 3. *There exist low noncomputable 2-c.e. degrees* $\mathbf{b} < \mathbf{a}$ *such that for any 2-c.e. degree* $\mathbf{v} \leq \mathbf{a}$ *either* $\mathbf{v} \leq \mathbf{b}$ *or* $\mathbf{b} \leq \mathbf{v}$.



Theorem 3 with Sacks's splitting theorem lead to the elementary difference of partial orders of low c.e. and low 2-c.e degrees. Moreover, since every 1-low degree is n-low for any $n > 1$ partial orders of n-low c.e. and n-low 2-c.e. degrees are not elementarily equivalent.

[Downey, Stob, 1993],[Downey, Yu, 2004] noticed that the question in the case of 2-low was open.

The following sentence φ shows that these partial orders are not elementarily equivalent.

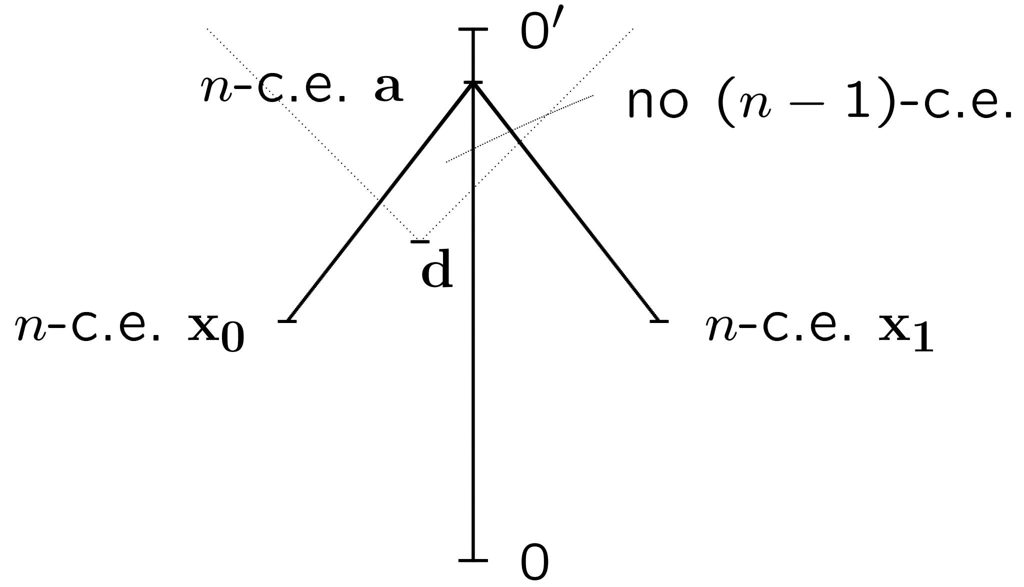
$$\varphi = \exists\, \mathbf{a},\ \mathbf{b}\, \forall\, \mathbf{v}\ (\mathbf{0} < \mathbf{b} < \mathbf{a}) \wedge [(\mathbf{v} \le \mathbf{a}) \longrightarrow (\mathbf{b} \le \mathbf{v}) \vee (\mathbf{v} \le \mathbf{b})].$$

[Faizrahmanov, 2008] in the case of 1-low c.e. and 1-low 2-c.e. degrees also get elementary difference. And another way to proof this result uses strongly noncuppability in 1-low c.e. degrees.

But these couldn't be applied immediately for the general case of n-low degrees.

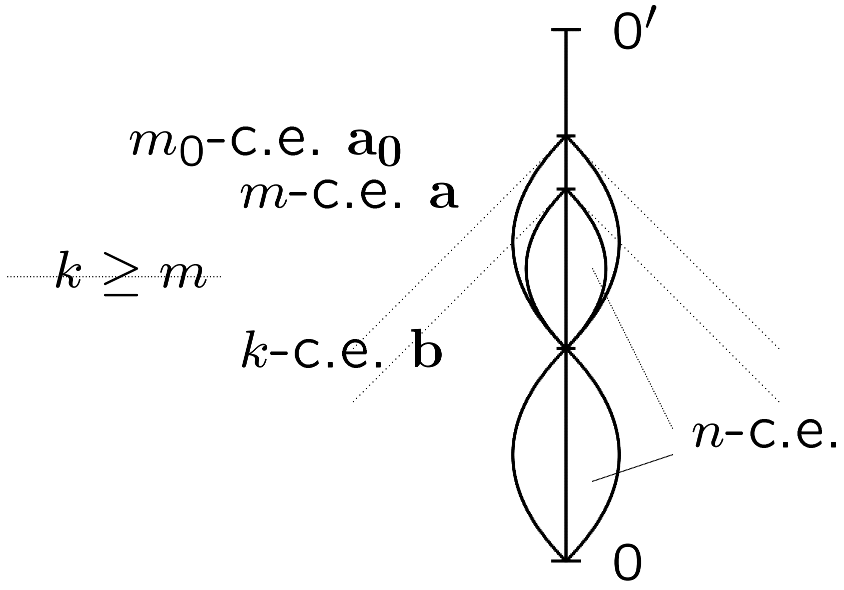
Some observation in n-c.e. degrees.

Theorem 4*. *Let* $\mathbf{a}$ *and* $\mathbf{d}$ *be properly* n*-c.e. and properly* k*-c.e. degrees, respectively, such that* $k \geq n$, $\mathbf{0} < \mathbf{d} < \mathbf{a}$ *and there are no* $(n-1)$*-c.e. degrees between* $\mathbf{a}$ *and* $\mathbf{d}$*. Then* $\mathbf{a}$ *is splittable avoiding upper cone of* $\mathbf{d}$*.*



Corollary 1*. *If* $\mathbf{b} < \mathbf{a_0}$ *are properly k-c.e. and properly m_0-c.e. degrees, respectively, and if they form "bubble" in n-c.e. degrees (for some $n \geq max(k, m_0)$) then $k < m_0$.*

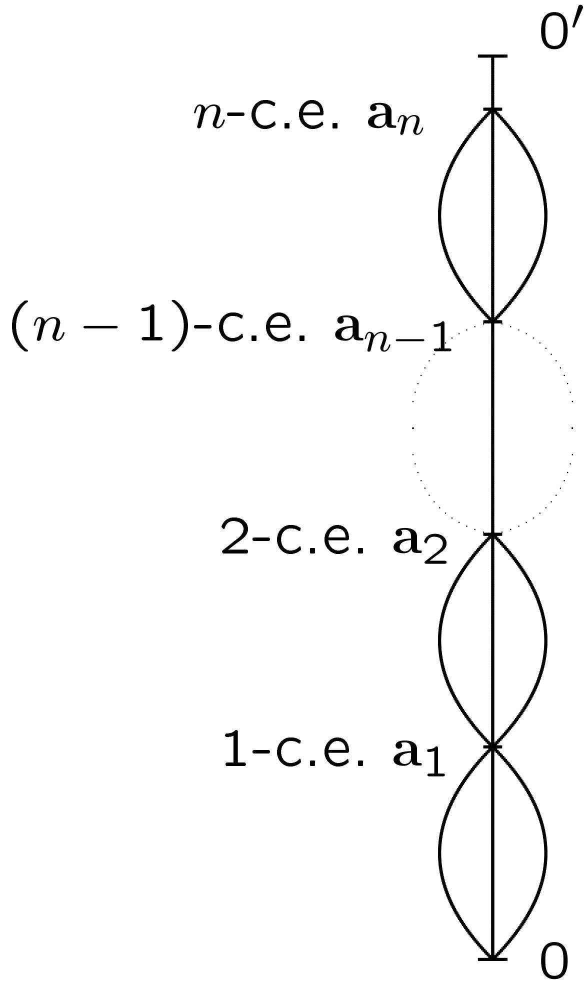
Proof. Every n-c.e. degree strictly between $\mathbf{b}$ and $\mathbf{a_0}$ also forms "bubble" with $\mathbf{b}$ in n-c.e. degrees. Clear, that there exist properly m-c.e. ($m \leq m_0$) degree $\mathbf{a}$ such that there no ($m-1$)-c.e. degrees between $\mathbf{b}$ and $\mathbf{a}$. So, if $k \geq m_0$ then $k \geq m$ and by Theorem 4* $\mathbf{a}$ is splittable in m-c.e. degrees avoiding upper cone of $\mathbf{b}$. Contradiction with the "bubble".



0′
m_0-c.e. $\mathbf{a_0}$
m-c.e. $\mathbf{a}$
$k \geq m$
k-c.e. $\mathbf{b}$
n-c.e.
0

Definition. Degrees $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ form "n-bubble" ($n > 2$) in a class of degrees $\mathcal{C}$ if $\mathbf{a}_i \in \mathcal{C}, (i = 1, ..., n)$, $\mathbf{0} < \mathbf{a}_1 < \mathbf{a}_2 < ... < \mathbf{a}_n$, the degrees $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_{n-1}$ form "$(n-1)$-bubble" and every degree from $\mathcal{C}$ and below $\mathbf{a}_n$ is comparable with $\mathbf{a}_{n-1}$.

By corollary 2* "n-bubbles" could be only of the following type.



Corollary 2*. There are no "$(n+1)$-bubbles" in n-c.e. degrees

Proof. Let $P(\mathbf{a})$ be a function such that $P(\mathbf{a}) = k$ where $\mathbf{a}$ is properly k-c.e. degree. If $\mathbf{a}_1$, $\mathbf{a}_2$, ...,$\mathbf{a}_{n+1}$ form "$(n+1)$-bubble" in n-c.e. degrees, then $P(\mathbf{a}_1) < P(\mathbf{a}_2) < ... < P(\mathbf{a}_{n+1}) \leq n$. This involves that $P(\mathbf{a}_1) \leq 0$. Contradiction.

Also we can see that "n-bubble" in n-c.e. degrees is unique (if it exists).

So, if such "n-bubble" exists and if Theorem 4* holds then we get that n-c.e. and m-c.e. degrees are not elementarily equivalent for any $n \neq m$.

Question. Does "n-bubble" exist in n-c.e. degrees?

THANK YOU FOR ATTENTION!