Forceless, ineffective, powerless proofs of descriptive set-theoretic dichotomy theorems

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Part I

Introduction

I. Introduction

A brief history

Set theory was born in 1873 with Cantor's realization that there is no injection of the reals numbers into the natural numbers.

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This came to be known as *Cantor's Continuum Hypothesis*, or CH, and the question of its truth appeared as Hilbert's first problem.

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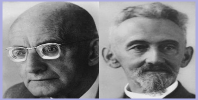
Today this subject is known as *descriptive set theory*. The first result in the area was actually established some time earlier:

Theorem (Cantor)

Suppose that X is a Polish space and $C \subseteq X$ is closed. Then exactly one of the following holds:

1. The set C is countable.
2. There is a perfect subset of C.

I. Introduction
A brief history



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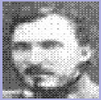

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Soon thereafter, Souslin further generalized Cantor's theorem to continuous images of functions from ω^ω to Hausdorff spaces, which he referred to as *analytic* sets.

Since then, the search for dichotomy theorems has played a fundamental role in the development of descriptive set theory.

I. Introduction

A brief history

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One was therefore led naturally to the belief that the abundance of such derivatives is the driving force underlying the great variety of dichotomy theorems in descriptive set theory.

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Theorem (Silver)

Suppose that X is a Hausdorff space and E is a co-analytic equivalence relation on X. Then exactly one of the following holds:

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Unlike earlier proofs, Silver's argument was a technical *tour de force* relying on a number of techniques from mathematical logic.

I. Introduction

A brief history

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Burgess combined Silver's theorem with his own reflection results to obtain an analogous fact for analytic equivalence relations.

Harrington-Shelah then found a direct proof of a much more general result using forcing and infinitary logic.

I. Introduction

A brief history

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While some of these were relatively straightforward generalizations of Silver's theorem, others relied on progressively more sophisticated and technically difficult refinements of Harrington's ideas.

I. Introduction

Motivating questions

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If not, is there another unifying explanation for these results?

I. Introduction
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Although our focus will be on Borel sets, the ideas behind our arguments should generalize to broader classes of definable sets.

Ideally we would like to isolate a common core from which all dichotomy theorems can be easily established.

I. Introduction

Towards a positive solution

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These theorems have classical proofs using nothing more than derivatives and the first separation theorem.

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This work is built upon the backbone of a family of dichotomy results which appear naturally in the study of colorings of definable graphs.

These theorems have classical proofs using nothing more than derivatives and the first separation theorem.

Moreover, these results can be combined with classical Baire category arguments to obtain many recent dichotomy theorems.

Part II

Chromatic numbers

II. Chromatic numbers

Basic definitions

Definition

A *graph* on X is an irreflexive symmetric set $\mathcal{G} \subseteq X \times X$.

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Definition

A set $B \subseteq X$ is *$\mathcal{G}$-discrete* if $\mathcal{G} \restriction B = \mathcal{G} \cap (B \times B) = \emptyset$.

II. Chromatic numbers

Basic definitions

Definition

A *homomorphism* from a graph $\mathcal{G}$ on X to a graph $\mathcal{H}$ on Y is a function $\phi\colon X \to Y$ with the property that

$$\forall x_0, x_1 \in X \; ((x_0, x_1) \in \mathcal{G} \implies (\phi(x_0), \phi(x_1)) \in \mathcal{H}).$$

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Definition

A *coloring* of $\mathcal{G}$ is a function $c \colon X \to I$ with the property that

$$\forall x_0, x_1 \in X \ ((x_0, x_1) \in \mathcal{G} \implies c(x_0) \neq c(x_1)).$$

II. Chromatic numbers

Basic definitions

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In particular, they isolated a $D_2(\mathbf{\Sigma}^0_1)$ acyclic graph on 2^ω which does not have a Baire measurable ω-coloring.

II. Chromatic numbers
Basic definitions

Fix sequences $s_n \in 2^n$ which are *dense* in the complete binary tree, in the sense that $\forall s \in 2^{<\omega} \exists n \in \omega \; (s \sqsubseteq s_n)$.

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Definition (Kechris-Solecki-Todorcevic)

Let $\mathcal{G}_0$ denote the graph on 2^ω consisting of all pairs of the form

$$(s_n{}^\frown i^\frown x, s_n{}^\frown (1-i)^\frown x),$$

where $i \in 2$, $n \in \omega$, and $x \in 2^\omega$.

Proposition (Kechris-Solecki-Todorcevic)

Suppose that $B \subseteq 2^\omega$ has the Baire property and is $\mathcal{G}_0$-discrete. Then B is meager.

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Theorem (Kechris-Solecki-Todorcevic)

Suppose that X is a Hausdorff space and $\mathcal{G}$ is an analytic graph on X. Then exactly one of the following holds:

1. There is a Borel ω-coloring of $\mathcal{G}$.
2. There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.

The Kechris-Solecki-Todorcevic argument uses the effective theory.

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However, it uses significantly less of the effective theory than the proofs of other results which at the time had no classical proofs.

II. Chromatic numbers

The $\mathcal{G}_0$ dichotomy

Observation

There is a proof of the $\mathcal{G}_0$ dichotomy which uses nothing more than a derivative and the first separation theorem.

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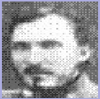
The $\mathcal{G}_0$ dichotomy can be combined with classical Baire category arguments so as to obtain many other dichotomy theorems.

Some results which follow from the $\mathcal{G}_0$ dichotomy



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5. Silver's theorem.
6. The Friedman-Harrington-Kechris generalization of Silver's theorem to quasi-metric spaces.
7. Louveau's characterization of the circumstances under which there is a Borel set which selects an F-class from each E-class, where E and F are equivalence relations and $[E : F] = 2$.

II. Chromatic numbers
An application

We will now give the first classical proof of:

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Suppose that X is a Hausdorff space and E is a co-analytic equivalence relation on X. Then exactly one of the following holds:

1. The equivalence relation E has only countably many classes.
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Proof of Silver's theorem (part 1 of 2)

Define $\mathcal{G} = E^c$.

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We can therefore assume that there is no such coloring.

Then there is a continuous homomorphism ϕ from $\mathcal{G}_0$ to $\mathcal{G}$.

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Mycielski's theorem gives a perfect set P of F-inequivalent points.

Then $\phi(P)$ is a perfect set of E-inequivalent points. ✓

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Observation

On the bright side, many are corollaries of natural generalizations.

Louveau has noticed that the $\mathcal{G}_0$ dichotomy generalizes to digraphs.

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One can even establish the digraph version of the $\mathcal{G}_0$ dichotomy from the original theorem and an easy Baire category argument.

II. Chromatic numbers
Simple generalizations

Some results which follow from the directed $\mathcal{G}_0$ dichotomy

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1. Louveau's generalization of Silver's theorem to quasi-orders which, in particular, gives a two-element basis for the class of uncountable co-analytic quasi-orders.

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1. Louveau's generalization of Silver's theorem to quasi-orders which, in particular, gives a two-element basis for the class of uncountable co-analytic quasi-orders.
2. The Friedman-Shelah characterization of separable linear quasi orders which, in particular, ensures that there are no co-analytic Souslin lines.

II. Chromatic numbers
Simple generalizations

Louveau has also noticed that the $\mathcal{G}_0$ dichotomy generalizes to n-dimensional hypergraphs.

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II. Chromatic numbers

Simple generalizations

Some results which follow from the n-dimensional $\mathcal{G}_0$ dichotomy

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1. Generalizations of the van Engelen-Kunen-Miller theorems characterizing subsets of vector spaces which are unions of countably many low-dimensional subspaces.

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2. The n-dimensional analog of Feng's special case of the open coloring axiom.

There are also somewhat more subtle generalizations of the $\mathcal{G}_0$ dichotomy to ω-length sequences of graphs.

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These also follow from straightforward modifications of the proof of the $\mathcal{G}_0$ dichotomy.

II. Chromatic numbers

Simple generalizations

Some results which follow from the sequential $\mathcal{G}_0$ dichotomy

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1. Hjorth's characterization of acyclic graphs having transversals.

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3. A characterization of real-valued functions of two variables which are sums of two real-valued functions of one variable.

Some results which follow from the sequential $\mathcal{G}_0$ dichotomy

1. Hjorth's characterization of acyclic graphs having transversals.
2. A characterization of countable-dimensional vector spaces.
3. A characterization of real-valued functions of two variables which are sums of two real-valued functions of one variable.
4. A characterization of real-valued cocycles on equivalence relations with invariant probability measures of a given type.

Part III

Local chromatic numbers

Definition

A *reduction* of an equivalence relation E on X to an equivalence relation F on Y is a function $\pi \colon X \to Y$ with the property that

$$\forall x_0, x_1 \in X \ (x_0 E x_1 \iff \pi(x_0) F \pi(x_1)).$$

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Definition

An equivalence relation is *smooth* if it is Borel reducible to $\Delta(2^\omega)$.

III. Local chromatic numbers

Basic definitions

Fix sequences $s_{2n} \in 2^{2n}$ which are dense in the complete binary tree.

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Definition

Let $\mathcal{G}_0^{\text{even}}$ denote the graph on 2^ω consisting of all pairs of the form

$$(s_{2n}{}^\frown i^\frown x, s_{2n}{}^\frown(1-i)^\frown x),$$

where $i \in 2$, $n \in \omega$, and $x \in 2^\omega$.

III. Local chromatic numbers
Basic definitions

Fix pairs $s_{2n+1} \in 2^{2n+1} \times 2^{2n+1}$ which are *dense* in the square of the complete binary tree, in the sense that

$$\forall s \in 2^{<\omega} \times 2^{<\omega} \exists n \in \omega \forall i \in 2 \; (s(i) \sqsubseteq s_{2n+1}(i)).$$

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Definition

Let $\mathcal{H}_0^{\mathrm{odd}}$ denote the graph on 2^ω consisting of all pairs of the form

$$(s_{2n+1}(i)^\frown i^\frown x, s_{2n+1}(1-i)^\frown (1-i)^\frown x),$$

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where $i \in 2$, $n \in \omega$, and $x \in 2^\omega$.

Definition

Let E_0^{odd} denote the smallest equivalence relation containing $\mathcal{H}_0^{\mathrm{odd}}$.

III. Local chromatic numbers

The local $\mathcal{G}_0$ dichotomy

Theorem

Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, and $\mathcal{G}$ is an analytic graph on X. Then exactly one of the following holds:

1. There is a smooth equivalence relation $F \supseteq E$ such that the graph $F \cap \mathcal{G}$ admits a Borel ω-coloring.
2. There is a continuous homomorphism $\pi \colon 2^\omega \to X$ from the pair $(\mathcal{G}_0^{\mathrm{even}}, E_0^{\mathrm{odd}})$ to the pair $(\mathcal{G}, E)$.

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Much as the $\mathcal{G}_0$ dichotomy yields a simple proof of Silver's theorem, the local $\mathcal{G}_0$ dichotomy yields a simple proof of the Harrington-Kechris-Louveau characterization of smooth equivalence relations.

Definition

Set $xE_0y \iff \exists m \in \omega \forall n \in \omega \setminus m \; (x(n) = y(n))$.

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Theorem

Suppose that X is a Hausdorff space, E is a co-analytic equivalence relation on X, and F is an analytic subequivalence relation of E. Then exactly one of the following holds:

1. There is a smooth equivalence relation between F and E.
2. There is a continuous embedding of (E_0, E_0) into (F, E).

III. Local chromatic numbers

The local $\mathcal{G}_0$ dichotomy

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Some results which follow from the generalized local $\mathcal{G}_0$ dichotomy

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Some results which follow from the generalized local $\mathcal{G}_0$ dichotomy

1. The Kanovei-Louveau theorem generalizing the Harrington-Kechris-Louveau theorem and the Harrington-Marker-Shelah characterization of linear quasi-orders.

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Some results which follow from the generalized local $\mathcal{G}_0$ dichotomy

1. The Kanovei-Louveau theorem generalizing the Harrington-Kechris-Louveau theorem and the Harrington-Marker-Shelah characterization of linear quasi-orders.
2. The Harrington-Marker-Shelah Borel Dilworth theorem.

Part IV

Broader notions of definability

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Motivation

Although the early descriptive set-theorists were primarily concerned with Borel sets, they did investigate whether their results could be pushed into the projective hierarchy.

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Many questions of this sort turned out to be independent.

IV. Broader notions of definability
Motivation

Definition

A non-empty set is κ-*Souslin* if it is the continuous image of κ^ω.

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Much as the standard axioms imply that a subset of an analytic Hausdorff space is Borel if and only if it is bi-analytic, appropriate determinacy axioms yield characterizations of many natural pointclasses in terms of κ-Souslin sets.

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Much as the standard axioms imply that a subset of an analytic Hausdorff space is Borel if and only if it is bi-analytic, appropriate determinacy axioms yield characterizations of many natural pointclasses in terms of κ-Souslin sets.

This suggests that one might try to understand such pointclasses by studying κ-Souslin sets in $\mathtt{ZF}$.

IV. Broader notions of definability

The $\mathcal{G}_0$ dichotomy revisited

A simplification of the classical proof of the $\mathcal{G}_0$ dichotomy gives:

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Theorem

Suppose that X is a Hausdorff space and $\mathcal{G}$ is a κ-Souslin graph on X. Then at least one of the following holds:

1. There is a κ-coloring of $\mathcal{G}$.
2. There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.

Definition

A set $B \subseteq X$ is *ω-universally Baire* if for every continuous function $\phi\colon \omega^\omega \to X$, the set $\phi^{-1}(B)$ has the Baire property.

Definition

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Theorem

Suppose that X is a Hausdorff space and E is a co-κ-Souslin equivalence relation on X which is ω-universally Baire. Then at least one of the following holds:

1. The equivalence relation E has at most κ-many classes.
2. There is a perfect set of pairwise E-inequivalent points.

IV. Broader notions of definability

The Harrington-Kechris-Louveau theorem revisited

The other graph-theoretic dichotomy theorems have similar generalizations to the κ-Souslin case.

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Theorem

Suppose that X is a Hausdorff space and E is a bi-κ-Souslin equivalence relation on X which is ω-universally Baire. Then at least one of the following holds:

1. There is a reduction of E to $\Delta(2^\kappa)$.
2. There is a continuous embedding of E_0 into E.

IV. Broader notions of definability

The Harrington-Kechris-Louveau theorem revisited

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Theorem (Kanovei)

Suppose that X is a Hausdorff space and $\mathcal{G}$ is a κ-Souslin graph on X. Then at least one of the following holds:

1. There is a κ^+-Borel κ-coloring of $\mathcal{G}$.
2. There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.

Part V

Conclusions

V. Conclusions

Advantages of the new techniques

The new ideas described here appear to be leading towards a classical explanation of descriptive set-theoretic dichotomy theorems.

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Unlike the approach from effective descriptive set theory, the arguments generalize to broader pointclasses.

Unlike the Harrington-Shelah-style forcing approach, the generalizations do not require a technical forcing hypothesis.

V. Conclusions

Advantages of the new techniques

Both in the Borel case and for the weaker versions of results in broader families of definable sets, the new proofs restore the intuition that the abundance of derivatives is at the heart of the matter.

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They also reveal an intermediate level of graph-theoretic dichotomies from which all others seem to follow.

V. Conclusions

Unfinished work

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Both these slides and drafts of lecture notes around this topic can be found at `http://glimmeffros.googlepages.com`.