Abstract Elementary Classes

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Outline

Introduction

S. Shelah introduced Abstract Elementary Classes (AECs) more than 25 years ago as a general framework for classification in non-elementary classes. More recently notable work has also been done by Baldwin, Grossberg, Hyttinen, Kesälä, Kolesnikov, Lessmann, and vanDieren, among others.

The first part of this talk is an introduction to the basic theory of AECs. We then present recent results which connect AECs to classical infinitary logics in unexpected ways.

AEC Axioms

Definition (Shelah)

An *Abstract Elementary Class* is a pair $(\mathbb{K}, \prec_{\mathbb{K}})$ where $\mathbb{K}$ is a class of L-structures for some vocabulary L and $\prec_{\mathbb{K}}$ is a binary relation on $\mathbb{K}$ (called $\mathbb{K}$-*substructure*) satisfying the following axioms:

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A1 (*closure under isomorphism*) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{N} \cong \mathcal{M}$ then $\mathcal{N} \in \mathbb{K}$; if $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ and $(\mathcal{N}, \mathcal{M}) \cong (\mathcal{N}', \mathcal{M}')$ then $\mathcal{M}' \prec_{\mathbb{K}} \mathcal{N}'$.

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A2 ($\prec_{\mathbb{K}}$ *is a strong substructure relation*) If $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ then $\mathcal{M} \subseteq \mathcal{N}$; if $\mathcal{M} \in \mathbb{K}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{M}$; if $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$ and $\mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{M}_2$ then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_2$.

AEC Axioms Continued

A3 (*existence of Löwenheim-Skolem number*) There is an infinite cardinal $LS(\mathbb{K})$ such that for every $\mathcal{M} \in \mathbb{K}$ and for every subset $A \subseteq \mathcal{M}$ there is some $\mathcal{M}' \prec_{\mathbb{K}} \mathcal{M}$ such that $A \subseteq \mathcal{M}'$ and $|\mathcal{M}'| \leq max\{|A|, LS(\mathbb{K})\}$. We also assume $LS(\mathbb{K}) \geq |L|$.

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A4 (*closure under unions of $\prec_{\mathbb{K}}$ chains*) Let $\{\mathcal{M}_i\}_{i \in \mu}$ be a $\prec_{\mathbb{K}}$-chain.

(a) $\bigcup_{i \in \mu} \mathcal{M}_i \in \mathbb{K}$

(b) For each $j \in \mu$, $\mathcal{M}_j \prec_{\mathbb{K}} \bigcup_{i \in \mu} \mathcal{M}_i$

(c) If $\mathcal{M}_i \prec_{\mathbb{K}} \mathcal{N}$ for all $i \in \mu$ then $\bigcup_{i \in \mu} \mathcal{M}_i \prec_{\mathbb{K}} \mathcal{N}$

Axioms Continued

A5 (*coherence*) If $\mathcal{M}_0, \mathcal{M}_1, \mathcal{N} \in \mathbb{K}$, $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{N}$, $\mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{N}$ and $\mathcal{M}_0 \subseteq \mathcal{M}_1$ then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$.

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Note that this definition is purely set-theoretic. In particular there is no "syntax", and neither $\mathbb{K}$ nor $\prec_{\mathbb{K}}$ is assumed to be defined in any way.

Examples of AECs

1. If $\mathbb{K} = Mod(T)$ for a first order theory T in a language L and $\prec_{\mathbb{K}}$ is elementary substructure, then $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC with $LS(\mathbb{K}) = |L|$.

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2. If $\mathbb{K} = Mod(T)$ for an $\forall\exists$ first order theory of L and $\prec_{\mathbb{K}}$ is substructure, then $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC with $LS(\mathbb{K}) = |L|$.

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Note that the same class $\mathbb{K}$ of structures may be an AEC under many different substructure relations, and the resulting AECs may have very different properties.

Infinitary Logics

- $L_{\omega_1,\omega}$ is the extension of first order logic allowing the conjunction and disjunction of countable sets of formulas.
- $L_{\infty,\omega}$ allows the conjunction and disjunction of arbitrary sets of formulas.

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In each case we restrict to formulas with just finitely many free variables.

- L_{∞,ω_1} allows conjunctions and disjunctions of arbitrary sets of formulas and the quantification of countable sequences of variables.
- L_{∞,κ^+} allows quantification of sequences of κ variables.

Examples Continued

3. If $\mathbb{K} = Mod(\sigma)$ where $\sigma \in L_{\omega_1,\omega}$ and $\prec_{\mathbb{K}}$ is elementary submodel with respect to a countable fragment containing σ then $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC with $LS(\mathbb{K}) = \omega$.

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4. If $\mathbb{K} = Mod(\sigma)$ for $\sigma \in L_{\omega_1,\omega}(Q)$ then $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC with $LS(\mathbb{K}) = \omega_1$ for an appropriate choice of $\prec_{\mathbb{K}}$.

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5. If $\mathbb{K} = Mod(\sigma)$ for $\sigma \in L_{\kappa^+,\omega}$ then $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC with $LS(\mathbb{K}) = \kappa$ where $\prec_{\mathbb{K}}$ is elementary submodel with respect to a fragment containing σ.

Non-Examples

6. If $\mathbb{K} = Mod(\sigma)$ for $\sigma \in L_{\omega_1,\omega_1}$ and $\prec_{\mathbb{K}}$ is elementary submodel with respect to L_{ω_1,ω_1} then $(\mathbb{K}, \prec_{\mathbb{K}})$ is usually *not* an AEC since $Mod(\sigma)$ is usually not closed under unions of *countable* L_{ω_1,ω_1}-elementary chains.

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7. Let L contain just the unary predicate P and let $\mathbb{K}$ be the class of L-structures $\mathcal{M}$ in which P and its complement have the same cardinality. Then there is *no* relation $\prec_{\mathbb{K}}$ such that $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC.

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   Why won't $\prec_{\mathbb{K}}$ defined on $\mathbb{K}$ by $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ iff $\mathcal{M} \subseteq \mathcal{N}$ and $|P^N - P^M| = |(\neg P^N) - (\neg P^M)|$ work?

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   Coherence fails!

$\mathbb{K}$-embeddings

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Definition

1. Let $\mathcal{M}, \mathcal{N} \in \mathbb{K}$. An embedding f of $\mathcal{M}$ into $\mathcal{N}$ is a $\mathbb{K}$-*embedding* if $f(\mathcal{M}) \prec_{\mathbb{K}} \mathcal{N}$.
2. $(\mathbb{K}, \prec_{\mathbb{K}})$ satisfies *joint embedding (JEP)* provided any two models in $\mathbb{K}$ can be $\mathbb{K}$-embedded into some model in $\mathbb{K}$.
3. $(\mathbb{K}, \prec_{\mathbb{K}})$ satisfies the *amalgamation property (AP)* if for any $\mathcal{M}, \mathcal{N}_0, \mathcal{N}_1 \in \mathbb{K}$ and any $\mathbb{K}$-embeddings f_0 and f_1 of $\mathcal{M}$ into $\mathcal{N}_0$ and $\mathcal{N}_1$ there are $\mathcal{M}^* \in \mathbb{K}$ and embeddings g_0 and g_1 of $\mathcal{N}_0$ and $\mathcal{N}_1$ into $\mathcal{M}^*$ such that $g_0(f_0(a)) = g_1(f_1(a))$ for all $a \in \mathcal{M}$.

Homogeneous Models

Theorem (Shelah)

Assume that $(\mathbb{K}, \prec_{\mathbb{K}})$ satisfies (AP), (JEP), and contains arbitrarily large models. Then for every $\kappa > LS(\mathbb{K})$ there is some $\mathcal{M} \in \mathbb{K}$ such that

(a) Every $\mathcal{N} \in \mathbb{K}$ of cardinality $< \kappa$ can be $\mathbb{K}$-embedded into $\mathcal{M}$.

(b) any isomorphism between $\mathbb{K}$-substructures of $\mathcal{M}$ of cardinality $< \kappa$ extends to an automorphism of $\mathcal{M}$.

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Such an $\mathcal{M}$ is *strongly* κ *model homogeneous*.

We abbreviate the hypothesis of (AP), (JEP), and arbitrarily large models by (AP+).

Galois Types

Assuming that $(\mathbb{K}, \prec_{\mathbb{K}})$ satisfies (AP+) we choose some sufficiently large μ and fix a strongly μ model homogeneous $\mathfrak{C} \in \mathbb{K}$, called the *Monster*, and consider just models $\mathcal{M}$ such that $\mathcal{M} \prec_{\mathbb{K}} \mathfrak{C}$ and $|\mathcal{M}| < \mu$.

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Definition (Shelah)

(AP+) Let $\mathcal{M} \prec_{\mathbb{K}} \mathfrak{C}$ with $|\mathcal{M}| < \mu$ and let $a \in \mathfrak{C}$.

(a) *$tp^g(a/\mathcal{M})$, the Galois type of a over $\mathcal{M}$*, is the orbit of a under all automorphisms of $\mathfrak{C}$ fixing $\mathcal{M}$ pointwise.

(b) $\mathcal{N} \prec_{\mathbb{K}} \mathfrak{C}$ *realizes $tp^g(a/\mathcal{M})$* if the orbit of a intersects $\mathcal{N}$.

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Use of the Monster is a convenience but is not necessary. Galois types can be defined using just (AP).

We will also consider Galois types over sets.

Galois Saturation

Galois saturated models may now be defined in the expected way.

Definition

(AP+) $\mathcal{M} \in \mathbb{K}$ is κ *Galois saturated* provided it realizes all Galois types over subsets of cardinality $< \kappa$.

For $\kappa > LS(\mathbb{K})$ it suffices to consider Galois types over $\mathbb{K}$-substructures of $\mathcal{M}$ of cardinality $< \kappa$.

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Assuming (AP+) every $\mathcal{M} \in \mathbb{K}$ has a κ Galois saturated $\mathbb{K}$-extension, and any two κ Galois saturated models of cardinality κ are isomorphic (if $\kappa > LS(\mathbb{K})$ or $cof(\kappa) = \omega$).

Galois Stability

Definition

(AP+) $(\mathbb{K}, \prec_{\mathbb{K}})$ is κ *Galois stable*, where $\kappa \geq LS(\mathbb{K})$, provided there are at most κ Galois types over every $\mathcal{M} \in \mathbb{K}$ of cardinality κ.

Theorem (Shelah)

(AP+) Assume that $(\mathbb{K}, \prec_{\mathbb{K}})$ is λ categorical where $\lambda > LS(\mathbb{K})$. Then $(\mathbb{K}, \prec_{\mathbb{K}})$ is κ Galois stable for all κ with $LS(\mathbb{K}) \leq \kappa < \lambda$.

Theorem (Shelah)

(AP+) Assume that $(\mathbb{K}, \prec_{\mathbb{K}})$ is κ Galois stable, where κ is regular and $\geq LS(\mathbb{K})$. Then $\mathbb{K}$ contains a κ Galois saturated model of cardinality κ.

Motivating Questions

Questions

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC.

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Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC.

1. What closure properties must $\mathbb{K}$ and $\prec_{\mathbb{K}}$ satisfy?
2. Is there a logic in which $\mathbb{K}$ and $\prec_{\mathbb{K}}$ can be defined?

Closure

Theorem

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Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \kappa$.

(a) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \equiv_{\infty,\kappa^+} \mathcal{N}$ then $\mathcal{N} \in \mathbb{K}$.

(b) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \prec_{\infty,\kappa^+} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

Definability

Corollary

Assume that $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC and $LS(\mathbb{K}) = \kappa$. Suppose that $\mathbb{K}$ contains at most λ models of cardinality λ for some λ with $\lambda^\kappa = \lambda$. Then $\mathbb{K} = Mod(\sigma)$ for some $\sigma \in L_{\infty,\kappa^+}$.
If $\mathbb{K}$ contains at most λ models of cardinality $\leq \lambda$ then we can take $\sigma \in L_{\lambda^+,\kappa^+}$.

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It suffices to assume just that $\mathbb{K}$ contains at most λ pairwise L_{∞,κ^+} elementarily inequivalent models of cardinality λ (or $\leq \lambda$).

Can We Do Better?

Question

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The answer is No, regardless of κ.

Example

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There is an AEC $(\mathbb{K}, \prec_{\mathbb{K}})$ with $LS(\mathbb{K}) = \omega$ which is κ categorical for all $\kappa \geq \omega$ and satisfies $(AP+)$ which is not closed under $L_{\infty,\omega}$ elementary equivalence.

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The vocabulary L contains just a unary predicate P.
$\mathbb{K}$ is the class of all $\mathcal{M}$ with $|P^{\mathcal{M}}| = \omega$ and $|\mathcal{M} - P^{\mathcal{M}}| \geq \omega$.

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$\mathbb{K}$ is the class of all $\mathcal{M}$ with $|P^{\mathcal{M}}| = \omega$ and $|\mathcal{M} - P^{\mathcal{M}}| \geq \omega$.
$\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ iff $\mathcal{M} \subseteq \mathcal{N}$ and $P^{\mathcal{M}} = P^{\mathcal{N}}$.

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What extra conditions on $(\mathbb{K}, \prec_{\mathbb{K}})$ with $LS(\mathbb{K}) = \kappa$ will guarantee closure under $L_{\infty,\kappa}$ elementary equivalence?

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What extra conditions on $(\mathbb{K}, \prec_{\mathbb{K}})$ with $LS(\mathbb{K}) = \kappa$ will guarantee closure under $L_{\infty,\kappa}$ elementary equivalence?

We answer this question for $\kappa = \omega$ and will discuss the general case.

Example Revisited

In the Example, $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ only holds if $P^{\mathcal{M}} = P^{\mathcal{N}}$. Thus whether or not $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ depends on how the *entire* set $P^{\mathcal{M}}$ is related to the *entire* set $P^{\mathcal{N}}$ — it is not a *local* property of $\mathcal{M}$ and $\mathcal{N}$.

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We need a condition which will imply that $\prec_{\mathbb{K}}$ has a local character.

Finite Character

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Definition

An AEC $(\mathbb{K}, \prec_{\mathbb{K}})$ has *finite character* provided that $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ whenever $\mathcal{M}, \mathcal{N} \in \mathbb{K}$, $\mathcal{M} \subseteq \mathcal{N}$, and for every tuple $\bar{a}$ from $\mathcal{M}$ there is a $\mathbb{K}$ embedding of $\mathcal{M}$ into $\mathcal{N}$ fixing $\bar{a}$.

Examples

1. If T is a first order theory then $(Mod(T), \prec)$ has finite character.
2. If $\sigma \in L_{\infty,\omega}$ then $(Mod(\sigma), \prec_{\infty,\omega})$ has finite character.

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2. If $\sigma \in L_{\infty,\omega}$ then $(Mod(\sigma), \prec_{\infty,\omega})$ has finite character.
3. The example given of an AEC not closed under $L_{\infty,\omega}$ elementary equivalence does *not* have finite character.

Closure

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ and having finite character.

(a) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ then $\mathcal{N} \in \mathbb{K}$.

(b) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \prec_{\infty,\omega} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

Definability

Corollary

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ and having finite character. Assume that $\mathbb{K}$ contains $\leq \lambda$ models of cardinality λ for some $\lambda \geq \omega$. Then $\mathbb{K} = Mod(\sigma)$ for some $\sigma \in L_{\infty,\omega}$.
If $\mathbb{K}$ contains $\leq \lambda$ models of cardinality $\leq \lambda$ then we can take $\sigma \in L_{\lambda^+,\omega}$.

It suffices to assume just that $\mathbb{K}$ contains $\leq \lambda$ pairwise $L_{\infty,\omega}$ elementarily inequivalent models of cardinality λ (or $\leq \lambda$).

Types and Saturation

We next look at similar closure and definability results for types and saturation.

Many of these results do not require finite character.

Closure for Galois Types

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ and satisfying (AP+). Let $\mathcal{M}, \mathcal{N} \in \mathbb{K}$, let $\bar{a}$ and $\bar{b}$ be tuples from $\mathcal{M}, \mathcal{N}$ respectively. Assume that $(\mathcal{M}, \bar{a}) \equiv_{\infty,\omega} (\mathcal{N}, \bar{b})$. Then $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$.

Note that we do not assume finite character.

Definability of Galois Types

Corollary

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$, satisfying (AP+) and having $\leq \lambda$ models of cardinality λ for some infinite λ. Then for every tuple $\bar{a}$ from $\mathfrak{C}$ there is $\varphi_{\bar{a}} \in L_{\infty,\omega}$ such that for every $\mathcal{N} \prec_{\mathbb{K}} \mathfrak{C}$ and every $\bar{b}$ from $\mathcal{N}$

$$\mathcal{N} \models \varphi_{\bar{a}}(\bar{b}) \text{ iff } tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset).$$

ω Saturated Models

Lemma

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC satisfying (AP+). Let $\mathcal{M} \in \mathbb{K}$. Then $\mathcal{M}$ is ω Galois saturated iff
for every $n \in \omega$, every $a_0, \ldots, a_{n-1} \in \mathcal{M}$, every $\mathcal{N} \in \mathbb{K}$ with $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$, and every $b_n \in \mathcal{N}$ there is some $a_n \in \mathcal{M}$ such that $tp^g(a_0, \ldots, a_n/\emptyset) = tp^g(a_0, \ldots, a_{n-1}, b_n/\emptyset)$.

Notation

(AP+) $\mathbb{K}^\omega$ is the class of all ω Galois saturated models in $\mathbb{K}$.

Closure of $\mathbb{K}^\omega$

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ satisfying (AP+).

(a) Let $\mathcal{M} \in \mathbb{K}^\omega$ and $\mathcal{N} \in \mathbb{K}$. Then $\mathcal{N} \in \mathbb{K}^\omega$ iff $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$.

(b) Let $\mathcal{M}, \mathcal{N} \in \mathbb{K}^\omega$ and let $\bar{a}, \bar{b}$ be tuples from $\mathcal{M}, \mathcal{N}$ respectively. Then $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$ iff $(\mathcal{M}, \bar{a}) \equiv_{\infty,\omega} (\mathcal{N}, \bar{b})$.

Characterizing Galois Types

Corollary

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ satisfying (AP+). Let $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ and let $\bar{a}$ and $\bar{b}$ be tuples from $\mathcal{M}, \mathcal{N}$ respectively. Then $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$ iff $(\mathcal{M}', \bar{a}) \equiv_{\infty,\omega} (\mathcal{N}', \bar{b})$ for some $\mathbb{K}$ extensions $\mathcal{M}'$ and $\mathcal{N}'$ of $\mathcal{M}, \mathcal{N}$ respectively.

Finitary AECs

Definition (Hyttinen-Kesälä)

An AEC $(\mathbb{K}, \prec_{\mathbb{K}})$ is *finitary* if $LS(\mathbb{K}) = \omega$, it has finite character, and it satisfies (AP+).

Definability of $\mathbb{K}^\omega$

Theorem

Let $(\mathbb{K}, \prec_\mathbb{K})$ be a finitary AEC.

(a) $\mathbb{K}^\omega = Mod(\sigma)$ for some complete $\sigma \in L_{\infty,\omega}$. If there is a countable model in $\mathbb{K}^\omega$ then $\sigma \in L_{\omega_1,\omega}$.

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(b) If $\mathcal{M}, \mathcal{N} \in \mathbb{K}^\omega$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ iff $\mathcal{M} \prec_{\infty,\omega} \mathcal{N}$. If there is a countable model in $\mathbb{K}^\omega$ then $\prec_{\infty,\omega}$ can be replaced by elementary submodel with respect to some countable fragment of $L_{\omega_1,\omega}$.

Categoricity

Lemma

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ which satisfies (AP+) and is λ categorical for some $\lambda \geq \omega$. Then every $\mathcal{M} \in \mathbb{K}$ with $|\mathcal{M}| \geq \lambda$ belongs to $\mathbb{K}^{\omega}$, and $\mathbb{K}^{\omega}$ contains a countable model.

Categoricity for Finitary AECs

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be a finitary AEC which is λ categorical for some $\lambda > \omega$. Then there is a complete sentence $\sigma \in L_{\omega_1,\omega}$ such that for every $\mathcal{M}$ with $|\mathcal{M}| \geq \lambda$, $\mathcal{M} \in \mathbb{K}$ iff $\mathcal{M} \models \sigma$.

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Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be a finitary AEC which is λ categorical for some $\lambda > \omega$. Then there is a complete sentence $\sigma \in L_{\omega_1,\omega}$ such that for every $\mathcal{M}$ with $|\mathcal{M}| \geq \lambda$, $\mathcal{M} \in \mathbb{K}$ iff $\mathcal{M} \models \sigma$.

Thus $\mathbb{K}$ is “almost” $L_{\omega_1,\omega}$ axiomatizible.

$\mathbb{K}^\omega$ as a Finitary AEC

Lemma

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ satisfying (AP+). Assume there is a countable model in $\mathbb{K}^\omega$ and that $\mathbb{K}^\omega$ satisfies the following:

if $\mathcal{M}, \mathcal{N} \in \mathbb{K}^\omega$ and $\mathcal{M} \prec_{\infty,\omega} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

Then $(\mathbb{K}^\omega, \prec_{\mathbb{K}})$ is a finitary AEC.

Types Over Models

Definition

(AP+) Let $\mathcal{M} \prec_{\mathbb{K}} \mathfrak{C}$, $\bar{a} \in \mathfrak{C}$. We define $tp_{\infty,\omega}(\bar{a}/\mathcal{M})$ to be $\{\varphi(\bar{x}, \bar{m}) : \varphi \in L_{\infty,\omega}, \bar{m} \in \mathcal{M}, \mathfrak{C} \models \varphi(\bar{a}, \bar{m})\}$.

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Using a theorem of Hyttinen and Kesälä we conclude:

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be finitary and ω Galois stable. Then for any countable $\mathcal{M} \in \mathbb{K}$ and tuples $\bar{a}, \bar{b}$, $tp^g(\bar{a}/\mathcal{M}) = tp^g(\bar{b}/\mathcal{M})$ iff $tp_{\infty,\omega}(\bar{a}/\mathcal{M}) = tp_{\infty,\omega}(\bar{b}/\mathcal{M})$.

Adding the assumption of “tameness” removes the cardinality restriction on $\mathcal{M}$.

Finite Character Revisited

We next look more closely at finite character.
What is the strength of this condition?

Characterization of Finite Character

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ which satisfies (AP). Then the following are equivalent:

(i) $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character.

(ii) For every $\mathcal{M}_0, \mathcal{M}, \mathcal{N} \in \mathbb{K}$ with $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ and $\mathcal{M}_0 \subseteq \mathcal{N}$, if $(\mathcal{M}, \bar{a}) \equiv_{\infty,\omega} (\mathcal{N}, \bar{a})$ for every tuple $\bar{a}$ from $\mathcal{M}_0$ then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{N}$.

Number of Finitary AECs

It is easy to see that there are exactly 2^{2^ω} AECs with Löwenheim-Skolem number ω.

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Theorem

There are exactly 2^{2^ω} finitary AECs.

Corollary

There are finitary AECs $(\mathbb{K}, \prec_{\mathbb{K}})$ where $\mathbb{K}$ is not axiomatizable by a sentence of $L_{\omega_1,\omega}$.

Finitary AEC Not Closed Under $\equiv_{\omega_1,\omega}$

Example

There is a finitary AEC $(\mathbb{K}, \prec_{\mathbb{K}})$ such that $\mathbb{K}$ is not closed under $L_{\omega_1,\omega}$ elementary equivalence. In addition $\mathbb{K}$ contains exactly ω_1 countable models, is κ Galois stable for all $\kappa > \omega$, but is not ω Galois stable.

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$\mathbb{K}$ is the class of all structures $\mathcal{M}$ for $L = \{P, <\}$ such that $<^{\mathcal{M}}$ holds only between elements of $P^{\mathcal{M}}$ and $(P^{\mathcal{M}}, <^{\mathcal{M}}) \cong (\alpha, <)$ for some ordinal $\alpha \leq \omega_1$. $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ holds iff $\mathcal{M} \subseteq \mathcal{N}$ and $(P^{\mathcal{M}}, <^{\mathcal{M}})$ is an initial segment of $(P^{\mathcal{N}}, <^{\mathcal{N}})$.

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Questions

- If $(\mathbb{K}, \prec_{\mathbb{K}})$ is finitary must $\mathbb{K} = Mod(\sigma)$ for some $\sigma \in L_{\infty,\omega}$?

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- If $(\mathbb{K}, \prec_{\mathbb{K}})$ is finitary must $\mathbb{K} = Mod(\sigma)$ for some $\sigma \in L_{\infty,\omega}$?
- If $(\mathbb{K}, \prec_{\mathbb{K}})$ is finitary and λ categorical for some $\lambda > \omega$ must $\mathbb{K} = Mod(\sigma)$ for some $\sigma \in L_{\omega_1,\omega}$?
  This will be true for $\lambda = \omega_1$ provided $\mathbb{K}$ contains only countably many countable models.

Finite Character When $LS(\mathbb{K}) > \omega$

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Example (G. Johnson)

For every regular $\kappa > \omega$ there is an AEC $(\mathbb{K}, \prec_{\mathbb{K}})$ with $LS(\mathbb{K}) = \kappa$, satisfying (AP+) and having finite character such that $\mathbb{K}$ is not closed under $L_{\infty,\kappa}$ elementary equivalence.

Finite Character When $LS(\mathbb{K}) > \omega$

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Theorem (G. Johnson)

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC having finite character and with $LS(\mathbb{K}) = \kappa$ where $cof(\kappa) = \omega$.

(a) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ then $\mathcal{N} \in \mathbb{K}$.

(b) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

Many of the other results using finite character also generalize from ω to cofinality ω.

$LS(\mathbb{K}) = \omega$ Without Finite Character

We next investigate in more detail AECs with $LS(\mathbb{K}) = \omega$ which do not have finite character.

In particular, we obtain stronger closure properties than hold for AECs with uncountable Löwenheim-Skolem number.

Closure Revisited

Recall the following special case of the first closure result.

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$.

(a) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \equiv_{\infty,\omega_1} \mathcal{N}$ then $\mathcal{N} \in \mathbb{K}$.

(b) If $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ and $\mathcal{M} \prec_{\infty,\omega_1} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

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(b) If $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ and $\mathcal{M} \prec_{\infty,\omega_1} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

But these results require only a small fragment of L_{∞,ω_1}.

Closure With Respect To $L^*_{\infty,\omega}$

Definition

$L^*_{\infty,\omega}$ is defined like $L_{\infty,\omega}$ except that formulas with ω many free variables are allowed. $\mathcal{M} \prec^*_{\infty,\omega} \mathcal{N}$ is defined using formulas of $L^*_{\infty,\omega}$.

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Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$.

(a) Let $\mathcal{M} \in \mathbb{K}$. Assume that for every $\varphi(\bar{x}, \bar{y}) \in L^*_{\infty,\omega}$ if $\mathcal{M} \models \forall\bar{x}\exists\bar{y}\varphi(\bar{x}, \bar{y})$ then $\mathcal{N} \models \forall\bar{x}\exists\bar{y}\varphi(\bar{x}, \bar{y})$. Then $\mathcal{N} \in \mathbb{K}$.

(b) If $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ and $\mathcal{M} \prec^*_{\infty,\omega} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

Categoricity

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ satisfying (AP+). Assume that $(\mathbb{K}, \prec_{\mathbb{K}})$ is λ categorical for some λ with $cof(\lambda) > \omega$. Then there is a complete $\sigma \in L_{\infty,\omega_1}$ such that $\mathbb{K}$ and $Mod(\sigma)$ contain precisely the same models of cardinality $\geq \lambda$. $Mod(\sigma) = \mathbb{K}^{\omega_1}$ contains a unique model of cardinality ω_1.

In particular, if $(\mathbb{K}, \prec_{\mathbb{K}})$ is ω_1 categorical then $Mod(\sigma)$ is the class of all uncountable models in $\mathbb{K}$.

Free Groups

The following is a special case of a more general result.

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$. Assume that $\mathbb{K}$ contains every free group. Then $\mathbb{K}$ contains every ω_1 free group (= group all of whose countable subgroups are free).

Relative Closure Under $\equiv_{\infty,\omega}$

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$. Let $\mathcal{M}_0, \mathcal{M}, \mathcal{N} \in \mathbb{K}$ with $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$. Let $\mathcal{N}_0 \subseteq \mathcal{N}$ and assume that $(\mathcal{M}, \mathcal{M}_0) \equiv_{\infty,\omega} (\mathcal{N}, \mathcal{N}_0)$. Then $\mathcal{N}_0 \in \mathbb{K}$ and $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}$.

Question

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ which does not have finite character. Is it possible for either or both of the following closure conditions to hold?

(a) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ then $\mathcal{N} \in \mathbb{K}$.

(b) If $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ and $\mathcal{M} \prec_{\infty,\omega} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

Some Proofs

We outline proofs of the following closure results.

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$.

(a) $\mathbb{K}$ is closed under $\equiv_{\infty,\omega_1}$.

(b) If $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character then $\mathbb{K}$ is closed under $\equiv_{\infty,\omega}$.

Some Proofs

We outline proofs of the following closure results.

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$.

(a) $\mathbb{K}$ is closed under $\equiv_{\infty,\omega_1}$.

(b) If $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character then $\mathbb{K}$ is closed under $\equiv_{\infty,\omega}$.

The main tool is *countable approximations*.

Countable Approximations

For any L structure $\mathcal{M}$ and any countable set s we define the *countable approximation* $\mathcal{M}^s$ as the substructure of $\mathcal{M}$ generated by $\mathcal{M} \cap s$.

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For any set C we define a countably complete filter $\mathcal{D}(C)$ on $\mathcal{P}_{\omega_1}(C)$ (=the set of countable subsets of C) as the set of all $X \subseteq \mathcal{P}_{\omega_1}(C)$ which contain some closed unbounded set (where *closed* means closed under unions of countable chains, and *unbounded* means contains an extension of every $s \in \mathcal{P}_{\omega_1}(C)$).

Almost All Countable s

Given one or more structures to be approximated we choose some set C which contains the universes of each structure, and we say that a property of their approximations holds *for almost all countable* s, abbreviated *a.e.*, provided it holds for all s in some $X \in \mathcal{D}(C)$.

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For example:

Theorem

$\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ iff $\mathcal{M}^s \cong \mathcal{N}^s$ a.e.

The Game $\mathcal{G}(X)$

For $X \subseteq \mathcal{P}_{\omega_1}(C)$ the game $\mathcal{G}(X)$ is defined as follows: at stage n player I chooses some $a_{2n} \in C$ and player II responds by choosing some $a_{2n+1} \in C$. Player II *wins* if $s = \{a_n : n \in \omega\} \in X$.

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Theorem

Player II has a winning strategy in $\mathcal{G}(X)$ iff $X \in \mathcal{D}(C)$.

Countable Approximations and $\prec_{\mathbb{K}}$

Lemma

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$.

(a) If $\mathcal{M} \in \mathbb{K}$ then $\mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$ a.e.

(b) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M}_0 \subseteq \mathcal{M}$ is countable then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ iff $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s$ a.e.

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(b) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M}_0 \subseteq \mathcal{M}$ is countable then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ iff $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s$ a.e.

Proof

(a) $\{s \in \mathcal{P}_{\omega_1}(\mathcal{M}) : \mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}\}$ is closed, by the Chains axiom, and unbounded, by $LS(\mathbb{K}) = \omega$.

(b) If $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ then $\mathcal{M}_0 \subseteq \mathcal{M}^s$ a.e., hence $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s$ a.e. by (a) and Coherence.

The Main Lemma

Main Lemma

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$. Let $\mathcal{M} \in \mathbb{K}$ and let $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ be countable. Let $\bar{a}$ be an ω sequence listing the elements of $\mathcal{M}_0$. Suppose that $(\mathcal{M}, \bar{a}) \equiv^*_{\infty,\omega} (\mathcal{N}, \bar{b})$ for some $\mathcal{N}$ and $\bar{b}$. Then $\bar{b}$ lists the elements of some $\mathcal{N}_0$ with $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ a.e.

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Proof (Outline)

Let $X = \{s \in \mathcal{P}_{\omega_1}(\mathcal{M}) : \mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}\}$, and let $Y = \{s \in \mathcal{P}_{\omega_1}(\mathcal{N}) : \mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s\}$. We use player II's winning strategy in $\mathcal{G}(X)$ and the back and forth properties of $\equiv^*_{\infty,\omega}$ to show that II has a winning strategy in $\mathcal{G}(Y)$.

The Theorem in the General Case

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$. Let $\mathcal{M} \in \mathbb{K}$ and assume $\mathcal{M} \equiv_{\infty,\omega_1} \mathcal{N}$. Then every countable subset of $\mathcal{N}$ is contained in some $\mathcal{N}_0$ such that $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ a.e. Hence $\mathcal{N} \in \mathbb{K}$ since it is the union of a family of countable models in $\mathbb{K}$ directed under $\prec_{\mathbb{K}}$.

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Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$. Let $\mathcal{M} \in \mathbb{K}$ and assume $\mathcal{M} \equiv_{\infty,\omega_1} \mathcal{N}$. Then every countable subset of $\mathcal{N}$ is contained in some $\mathcal{N}_0$ such that $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ a.e. Hence $\mathcal{N} \in \mathbb{K}$ since it is the union of a family of countable models in $\mathbb{K}$ directed under $\prec_{\mathbb{K}}$.

Proof (Outline)

Let $B_0 \subseteq \mathcal{N}$ be countable. We find ω sequences $\bar{a}$ listing the elements of some $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ and $\bar{b}$ listing the elements of some $\mathcal{N}_0$ containing B_0 such that $(\mathcal{M}, \bar{a}) \equiv^*_{\infty,\omega_1} (\mathcal{N}, \bar{b})$. By the Main Lemma we conclude that $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ a.e.

Main Lemma for Finite Character Case

Main Lemma

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ having finite character. Let $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ be countable. Let $\bar{a}$ be an ω sequence listing the elements of $\mathcal{M}_0$. Suppose that $(\mathcal{M}, \bar{a}) \equiv_{\infty,\omega} (\mathcal{N}, \bar{b})$ for some $\mathcal{N}$ and $\bar{b}$. Then $\bar{b}$ lists the elements of some $\mathcal{N}_0$ with $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^S$ a.e.

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Proof (Outline)

For each $n \in \omega$ let $Y_n = \{s :$ there is a $\mathbb{K}$ embedding of $\mathcal{N}_0$ into $\mathcal{N}^s$ fixing $b_0, \ldots, b_n\}$. We show II has a winning strategy in every $\mathcal{G}(Y_n)$ as before. Therefore $Y = \bigcap Y_n \in \mathcal{D}(\mathcal{N})$ by countable completeness, and $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ for all $s \in Y$ by finite character.

Theorem in the Finite Character Case

Theorem

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$ and having finite character. Let $\mathcal{M} \in \mathbb{K}$ and assume $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$. Then every countable subset of $\mathcal{N}$ is contained in some $\mathcal{N}_0$ such that $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ a.e. Hence $\mathcal{N} \in \mathbb{K}$ since it is the union of a family of countable models in $\mathbb{K}$ directed under $\prec_{\mathbb{K}}$.

A Strong Closure Property

Definition

A class $\mathbb{K}$ is *closed* provided that $\mathcal{M} \in \mathbb{K}$ iff $\mathcal{M}^s \in \mathbb{K}$ a.e.

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Facts

1. If $\sigma \in L_{\omega_1,\omega}$ then $Mod(\sigma)$ is closed.
2. If $\mathbb{K}$ is closed then $\mathbb{K}$ is closed under $\equiv_{\infty,\omega}$.
3. If $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC with $LS(\mathbb{K}) = \omega$ then $\mathcal{M} \in \mathbb{K}$ implies $\mathcal{M}^s \in \mathbb{K}$ a.e.
4. If $\mathbb{K}$ is closed it is determined completely by the collection of countable models in $\mathbb{K}$.

Question and Lemma

Question

What conditions on an AEC $(\mathbb{K}, \prec_{\mathbb{K}})$ with $LS(\mathbb{K}) = \omega$ will imply that $\mathbb{K}$ is closed? In particular if $(\mathbb{K}, \prec_{\mathbb{K}})$ is finitary and λ categorical for some uncountable λ must $\mathbb{K}$ be closed?

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Lemma

Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC with $LS(\mathbb{K}) = \omega$. Assume that $\mathcal{M} \prec_{L^*} \mathcal{N}$ implies $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ for all $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ for some countable fragment L^* of $L_{\omega_1,\omega}$. Then $\mathbb{K}$ is closed.

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