on local stability and more

Alf Onshuus

1 August 2009

- We will work inside a monster (sufficiently saturated) model $\mathcal{C}$ of a first order theory T, containing all sets and tuples that we will mention.
- By $\models \phi(a, b)$ we will mean that $\mathcal{C} \models \phi(a, b)$
- We will work in $\mathcal{C}^{eq}$.
- By "definable" we will usually mean definable with parameters.

Stable theories

Stable theories can be characterize by:

- forking works well (transitivity, symmetry, etc.) and
- lots of invariance:
  - non forking extensions are rigid (finite orbit or, if the original type was over algebraically closed sets, unique).
  - types are definable.

Stable theories

Stable theories can be characterize by:

- forking works well (transitivity, symmetry, etc.) and
- lots of invariance:
  - non forking extensions are rigid (finite orbit or, if the original type was over algebraically closed sets, unique).
  - types are definable.

Simple theories are those for which forking works well (in fact, either symmetry or transitivity of forking implies the theory is simple).

Dependent theories preserve some of the rigidity for types that we had in stable theories.

Definition:

- A formula $\phi(x, y)$ has the *order property* if it is consistent with T to have sequences $\langle a_i \rangle$ and $\langle b_i \rangle$ such that $\models \phi(a_i, b_j)$ if and only if $i \leq j$.
- A structure M is *stable* if the order property cannot be witnessed with elements in M.
- A theory is *stable* if every model of T is stable.

Pillay ("local stability theory" v.1): When one restricts the language to formulas without the order property, one gets all the results of stability theory for the resulting reduct.

Pillay (“local stability theory” v.1): When one restricts the language to formulas without the order property, one gets all the results of stability theory for the resulting reduct.

Note: “local” in the title means we will look at properties of particular sets and types within a structure, as opposed the the Pillay (?) version of looking at restrictions of the language.

Definition: A structure M is *o-minimal* if there is a binary relation $< \in \mathcal{L}(M)$ such that the interpretation of $<$ in M is a total linear order and every definable subset of M is a finite union of $<$-intervals and points.

Good properties of types in stable theories

- A good notion of independence, “size” and invariant extensions: non forking.
- Types over algebraically closed sets have unique non forking extensions.
- Types are definable.
- All substructures are stably embedded (which means all the structure can be defined internally).
- Types have a small number of extensions.

Definition of a stable type $p(x)$.

Let $\phi(x, y)$ be a formula. Then ϕ does not witness the order property in p if:

Definition of a stable type $p(x)$.

Let $\phi(x, y)$ be a formula. Then ϕ does not witness the order property in p if:

- The are no sequences $\langle a_i \rangle$ and $\langle b_j \rangle$ of elements realizing $p(x)$ such that $\models \phi(a_i, b_j)$ if and only if $i < j$.

This definition does not imply any of the nice properties that types have in stable theories

or

Definition of a stable type $p(x)$.

Let $\phi(x, y)$ be a formula. Then ϕ does not witness the order property in p if:

• The are no sequences $\langle a_i \rangle$ and $\langle b_j \rangle$ of elements realizing $p(x)$ such that $\models \phi(a_i, b_j)$ if and only if $i < j$.

This definition does not imply any of the nice properties that types have in stable theories

or

• (Lascar-Poizat) The are no sequences $\langle a_i \rangle$ of elements realizing $p(x)$ and $\langle b_j \rangle$ such that $\models \phi(a_i, b_j)$ if and only if $i < j$.

If we want to imitate stability theory within realizations of a type, this is the right definition.

Equivalent definitions

(Lascar-Poizat)

• A type $p(x)$ is stable if and only if it has a small number of extensions.

• A type $p(x)$ is stable if and only if for every formula $\phi(x, y)$ there is some $\theta(x) \in p(x)$ such that $\theta(x) \wedge \phi(x, y)$ has the order property.

Properties of stable types

- (Corollary) Stability is preserved under extensions, and concatenation of tuples.

Properties of stable types

• (Corollary) Stability is preserved under extensions, and concatenation of tuples.

Forking behaves nicely:

• Stable types over algebraically closed sets have unique non forking extensions.

• Non forking has transitivity and symmetry whenever we have stable types on *one side* of the independence relation.

Properties of stable types

• (Corollary) Stability is preserved under extensions, and concatenation of tuples.

Forking behaves nicely:

• Stable types over algebraically closed sets have unique non forking extensions.

• Non forking has transitivity and symmetry whenever we have stable types on *one side* of the independence relation.

• Stable types are definable, existence of a stable canonical base.

Are stable types closed under non forking restrictions?

• A formula $\theta(x)$ defines a (weakly) stable subset in M^n if $\theta(M^n)$ with all the inherited structure of M is stable.
• ("Stable and stably embedded") X is stably embedded in M if every M-definable subset of X is definable with parameters from X.

Borrowing the definition for stable subsets.

A formula $\theta(x)$ defines a stable set if for any $\phi(x, y)$ the formula $\theta(x) \wedge \phi(x, y)$ does not have the order property.

Equivalently, a definable subset X of a model M is stable if given any formula $\phi(x, y)$ then for some n we cannot find a sequence $\langle a_i \rangle_{i \leq n}$ and $\langle b_i \rangle_{i \leq n}$ such that $\phi(a_i, b_j)$ if and only if $i < j$.

This notion allows us to use a lot of the stability theoretic tools AND is equivalent to the "stable and stably embedded" notion defined above.

Proof.

This notion allows us to use a lot of the stability theoretic tools AND is equivalent to the "stable and stably embedded" notion defined above.

Proof.

• Let $\theta(x)$ define a stable set (assume for simplicity that θ has no parameters), let $b \in M$ and consider the set $Y := \theta(M^n) \wedge \phi(M^n, b)$.

This notion allows us to use a lot of the stability theoretic tools AND is equivalent to the "stable and stably embedded" notion defined above.

Proof.

• Let $\theta(x)$ define a stable set (assume for simplicity that θ has no parameters), let $b \in M$ and consider the set
$Y := \theta(M^n) \wedge \phi(M^n, b)$.
• If $\psi(y, x) := \theta(x) \wedge \phi(x, y)$ then $\psi(y, x)$ has the order property.

This notion allows us to use a lot of the stability theoretic tools AND is equivalent to the "stable and stably embedded" notion defined above.

Proof.

• Let $\theta(x)$ define a stable set (assume for simplicity that θ has no parameters), let $b \in M$ and consider the set $Y := \theta(M^n) \wedge \phi(M^n, b)$.
• If $\psi(y, x) := \theta(x) \wedge \phi(x, y)$ then $\psi(y, x)$ has the order property.
• The ψ-definition of $tp(b/\theta(M^n))$ is definable over $\theta(M^n)$ and by definition the elements in M satisfying this definition must be precisely Y. □

Results in geometric stability theory.

(See Hasson-O.)

• (gen. Hrushovski) If a stable minimal type is locally modular and not trivial, then one can define a group that "governs" the algebraic closure.

• (gen. Buechler) If a stable type is non trivial, then it is contained in a definable stable set.

• (gen. Buechler) If a stable type is not locally modular, then it is contained in a definable strongly minimal set.

• In group G definable in a *superrosy* theory there is a stable subgroup H such that the quotient G/H does not have any definable stable subsets (in fact, it contains no stable types).

First applications: expanding o-minimality.

(Joint work with Assaf Hasson)

Questions.

- Can we somehow characterize theories interpretable in o-minimal structures (i.e. find a nice enough superclass)?

some applications in structures interpretable in o-minimal theories

Theorem

In structures interpretable in o-minimal theories þ-minimal types are either stable or "locally definably isomorphic" to the original structure.

Corollary

If G is interpretable in an o-minimal group, one can find a stable subgroup H such that every minimal type in G/H is "locally isomorphic" to the original o-minimal order.

Corollary

If G is interpretable in an o-minimal group, one can find a stable subgroup H such that every minimal type in G/H is "locally isomorphic" to the original o-minimal order.

Can one conclude that G/H can be "coordinatized" by types which are locally isomorphic to the original o-minimal structure?

Another geometric stability theory slide

• One has a good geometric understanding of (þ-)minimal types:

(gen. Peterzil-Starchenko) Unstable þ-minimal types have a Zilber-type trichotomy theorem.

So we have a full Zilber-type trichotomy theorem modulo Zilber's conjecture for stable types definable in o-minimal structures.

Second set of applications: When stable types can have implications over the whole structure.

(We will now work mostly within a dependent theory.)
(Example and first framework: Haskell, Hrushovski, Macpherson)
(Joint work with Alex Usvyatsov.)

We work in a 3-sorted structure

$$((K, +, *), (\Gamma, +), (k, +, *), v, res)$$

where $v : K \to \Gamma$ is the valuation map and res is the map into the residue field.

Hrushovski, Haskell and Macpherson (HHM) proved that in algebraically closed valued fields (ACVF), the understanding of stable-like types has consequences in the whole structure.

HHM prove that in ACVF there were types called *stably dominated types* which were determined by stable sets in a significant way.

and

HHM prove that in ACVF there were types called *stably dominated types* which were determined by stable sets in a significant way.

and

They proved that one could analyze any type in ACVF using a definable family of stably dominated types.

HHM's definition

A type $tp(A/C)$ in a dependent theory T is *stably dominated* if there is a function C-definable function f that takes A into a subset A^{st} of a stable ("stable and stably embedded") C-definable set such that given any D if $A^{st} \downarrow_C D$ then

$$tp(D/A^{st}) \vdash tp(D/A).$$

HHM's definition 2

All the results they needed for stably dominated types hold for the following generalization:
A type $tp(A/C)$ in a dependent theory T is *stably dominated* if there is a function C-definable function f such that $tp(f(A)/C)$ is stable and such that given any D if $f(A) \downarrow_C D$ then

$$tp(D/f(A)) \vdash tp(D/A).$$

This definition works very well in ACVF's, but the definition seems to make it an object which is not close to standard stability-theoretic objects.

domination

We will say that a dominates b over C if given any D, if $a \mathop{\smile\!\!\!\!\!\!\;|}_C D$ then $b \mathop{\smile\!\!\!\!\!\!\;|}_C D$.

Theorem (T dependent)

Given a, b and C, if $tp(b/C)$ is stable and b dominates a over C, then $tp(a/C)$ is stably dominated (by $Cb(tp(b/Ca))$).

Proof.

Theorem (T dependent)

Given a, b and C, if $tp(b/C)$ is stable and b dominates a over C, then $tp(a/C)$ is stably dominated (by $Cb(tp(b/Ca))$).

Proof.

One direction follows from the definitions. For the other direction we have two key steps.

Theorem (T dependent)

Given a, b and C, if $tp(b/C)$ is stable and b dominates a over C, then $tp(a/C)$ is stably dominated (by $Cb(tp(b/Ca))$).

Proof.

One direction follows from the definitions. For the other direction we have two key steps.

• Show that if a type is $tp(a/C)$ dominated by a stable type then it has one (or "few" if $C \neq \mathrm{acl}(C)$) non forking extensions.

Theorem (T dependent)

Given a, b and C, if $tp(b/C)$ is stable and b dominates a over C, then $tp(a/C)$ is stably dominated (by $Cb(tp(b/Ca))$).

Proof.

One direction follows from the definitions. For the other direction we have two key steps.

• Show that if a type is $tp(a/C)$ dominated by a stable type then it has one (or "few" if $C \neq \mathrm{acl}(C)$) non forking extensions.

•. Stable domination follows using the above, automorphisms and some forking calculus. □

Weight

Definition: The *preweight* of a type $p(x) = tp(a/C)$ is the supremum of the set of cardinals λ for which there is an C-independent set $\{b_i \mid i < \lambda\}$ such that $a \not\perp_C b_i$ for all i.

Weight

Definition: The *preweight* of a type $p(x) = tp(a/C)$ is the supremum of the set of cardinals λ for which there is an C-independent set $\{b_i \mid i < \lambda\}$ such that $a \not\perp_C b_i$ for all i.

Definition: The *stable preweight* of a type $p(x) = tp(a/C)$ is the supremum of the set of cardinals λ for which there is an C-independent set $\{b_i \mid i < \lambda\}$ such that $tp(b_i/C)$ is stable and $a \not\perp_C b_i$ for all i.

For stable preweight we can choose the witnesses to satisfy stable types.

Theorem

In theories of finite weight, if the weight of a tuple is equal to the stable weight, then the tuple is stably dominated. Even more, if $\{b_i \mid i < n\}$ is a C-independent set such that $tp(b_i/A)$ is stable and $a \not\perp_C b_i$ for all i and the weight of a over C is n, then $b_0, \ldots, b_{n-1}$ dominates a over C.

In many ways, the stable weight will give precisely how close a type comes to being stably dominated and it will provide with witnesses for this.

local o-minimality

O-minimality depends so much on definable sets that there is little hope to have a "strong" definition for o-minimal types.

Definition: Given an ω-saturated model M, a formula $\theta(x)$ will be said to define an *o-minimal definable subset of M* if there is an M-definable order $<$ with domain $\theta(\mathcal{C})$ such that every definable M-definable subset of $\theta(M)$ is a finite union of $<$-intervals and points.

• ω-saturation is necessary but other than that defining an o-minimal subset depends only on θ (i.e. is closed under extensions of M).

• ω-saturation is necessary but other than that defining an o-minimal subset depends only on θ (i.e. is closed under extensions of M).

All the properties of o-minimality are true in $\theta(M)$ and $\theta(M)$ is stably embedded in M.

What about local dependence?

Definition:

- A formula $\phi(x, y)$ has the *independence property* if for any n it is consistent with T to have sequences $\langle a_i \rangle_{i \leq n}$ and $\langle b_I \rangle_{I \subset \mathbb{N}}$ such that $\models \phi(a_i, b_I)$ if and only if $i \in I$.
- A theory is *dependent* if no formula has the independence property.

What about local dependence?

Definition:

- A formula $\phi(x, y)$ has the *independence property* if for any n it is consistent with T to have sequences $\langle a_i \rangle_{i \leq n}$ and $\langle b_I \rangle_{I \subset \mathbb{N}}$ such that $\models \phi(a_i, b_I)$ if and only if $i \in I$.
- A theory is *dependent* if no formula has the independence property.

Definition: A type $p(x)$ will have the independence property if there is a sequence $\langle a_i \rangle_{i \in \omega}$ such that for every I subset of ω there is some b_I such that $\models \phi(a_i, b_I)$ if and only if $i \in I$.

(gen Shelah 783) If we make a dependent type definable (adding externally definable subsets) do we get a dependent theory?