II.

Recall: In the K^c construction, we recursively produce a sequence of models $\mathcal{M}_\xi$ and $\mathcal{N}_\xi$.

At successor stages, we may add new certified extenders:

If $\mathcal{M}_\xi = (|\mathcal{M}_\xi|; \in, \vec{E}, \emptyset)$ is defined and there is a certified F which may be added to $\mathcal{M}_\xi$, then in fact we let $\mathcal{N}_{\xi+1} = (|\mathcal{M}_\xi|; \in, \vec{E}, F)$.

If we'll never add any extenders, then the $\mathcal{M}_\xi$'s and $\mathcal{N}_\xi$'s will always just be initial segments of L.

We were facing four problems:

(1) In the case where we add the next certified F, is F unique?

(2) $\xi \mapsto \mathcal{M}_\xi \cap OR$ is not monotone. Given α, is there some ξ_α s.t. $\mathcal{M}_\xi$ agrees with $\mathcal{M}_{\xi_\alpha}$ up thru α for all $\xi \geq \xi_\alpha$?

(3) Can we prove statements about the models $\mathcal{M}_\xi$ as we did for L (for instance GCH, $\square_\kappa$, etc.)?

(4) Can we prove Covering?

We'll deal with (1) — (3) first, then with (4).

Common theme behind all these issues:

<u>Iterability</u>.

We would like the models $\mathcal{N}_\xi$ from the K^c construction to be iterable in much the same way as $0^\#$ is iterable.

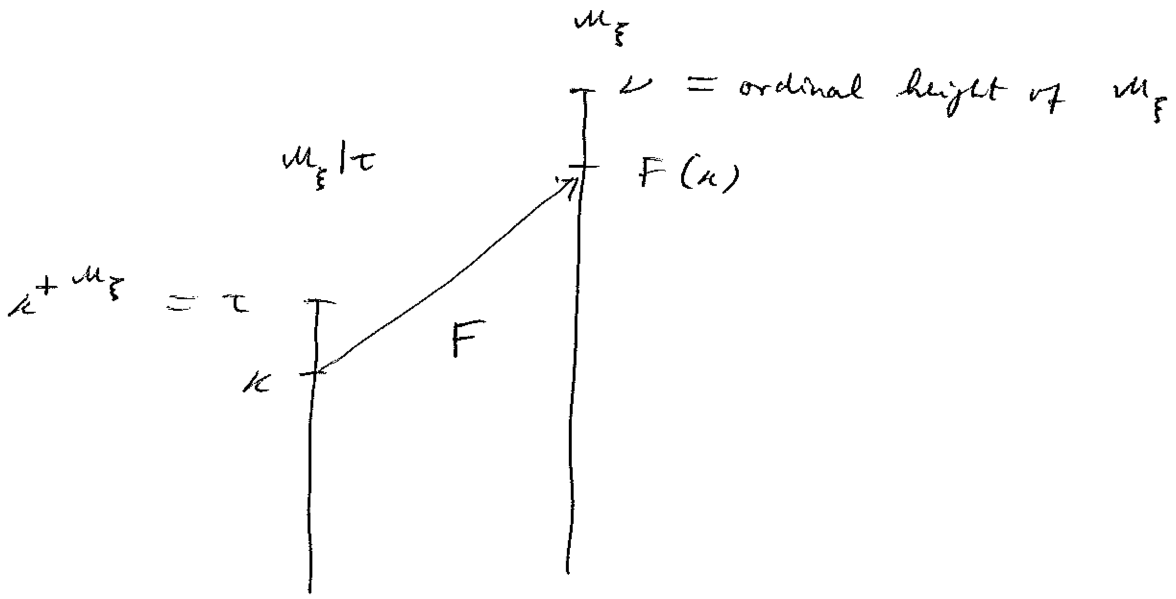
Basically,

$$\mathcal{N}_\xi = (|\mathcal{N}_\xi|; \in, (E_\alpha : \alpha \leq \hat{\xi})),$$

where either $E_\alpha = \emptyset$ or else E_α is an extender over $\mathcal{N}_\xi | \alpha$ = the model $\mathcal{N}_\xi$, cut off at α, for all α.

(Here, $\hat{\xi}$ is the ordinal height of $\mathcal{N}_\xi$.)

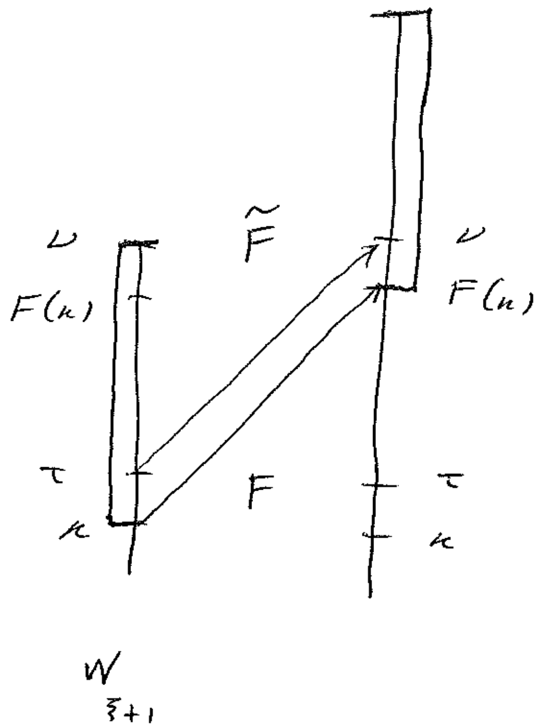
If we add a certified extender F to $\mathcal{M}_\xi = (|\mathcal{M}_\xi| ; \in, \vec{E}, \emptyset)$ to obtain $\mathcal{N}_{\xi+1} = (|\mathcal{M}_\xi| ; \in, \vec{E}, F)$, then,



exactly as in the case of $0^{\#}$, Setting $\kappa = \mathrm{crit}(F)$ and $\tau = \kappa^{+\mathcal{M}_\xi}$,

F is a map from $\mathcal{M}_\xi | \tau$ to $\mathcal{M}_\xi$ s.t. $F(\kappa)$ is the largest cardinal of $\mathcal{M}_\xi$ and the map F is Σ_0 - cofinal.

Exactly as before, we may extend F to act on the entire $\mathcal{M}_\xi$ (i.e., also on $\mathcal{W}_{\xi+1}$), getting $\tilde{F}$:

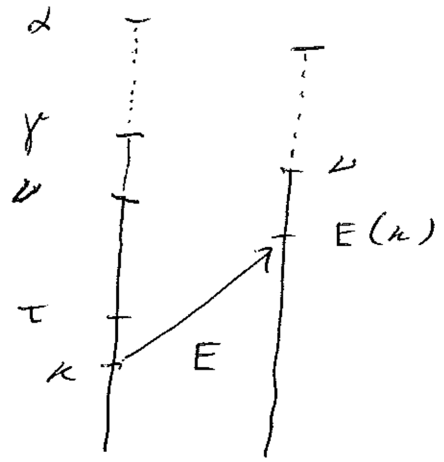


We shall write $\mathrm{Ult}(\mathcal{W}_{\xi+1}; F)$ to denote the target model of $\tilde{F}$.

$\mathrm{Ult}(\mathcal{W}_{\xi+1}; F) = (|\tilde{\mathcal{M}}|; \in, \vec{\tilde{E}}, F^*)$, where $\vec{\tilde{E}} \restriction \nu = \vec{E} \restriction \nu$ (<u>coherency</u>), $\tilde{E}_\nu = \emptyset$, and $\mathrm{crit}(F^*) = F(\kappa)$.

Now let E be <u>any</u> <u>one</u> of the extenders from $\vec{E}^\frown F$, where $\mathcal{W}_\xi = (|\mathcal{W}_\xi|; \in, \vec{E}, F)$, some ξ.

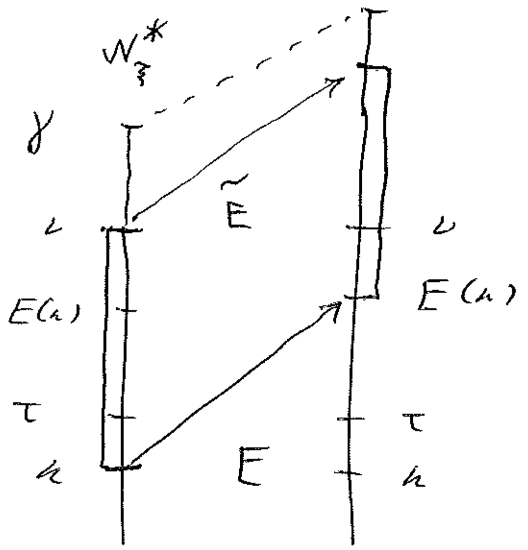
Let α = the ordinal height of $\mathcal{W}_\xi$, and let $\gamma \le \alpha$ be maximal such that E measures all the subsets of $\text{crit}(E)$ which are in $\mathcal{W}_\xi | \gamma$.



If $\vec{E}^\frown F = (E_\beta : \beta \le \alpha)$ and $E = E_\nu$, then $\nu \le \gamma \le \alpha$.

We shall have, setting $\kappa = \mathrm{crit}(E)$, $\tau = \kappa^{+W_\xi|\nu}$, that $\tau = \kappa^{+W_\xi|\gamma}$.

We may then extend the map E to a map $\tilde{E}$ which acts on $W_\xi|\gamma$ in much the same way as F was extended to $\tilde{F}$ above: $\mathrm{Ult}(W_\xi^*;E)$



We shall write W_ξ^* for $W_\xi|\gamma$, and we shall write $\mathrm{Ult}(W_\xi^*;E)$ for the target model of $\tilde{E}$.

Again, $\mathrm{Ult}(W_\xi^*;E) = (|\tilde{m}|;\in,\vec{\tilde{E}},F^*)$, where $\vec{\tilde{E}}\upharpoonright\nu = \vec{E}\upharpoonright\nu$, (coherency) $\vec{\tilde{E}}_\nu = \emptyset$.

We may now start iterating the models W_ξ.

By a *premouse* we shall mean a model

$$\mathcal{N} = (|W|;\ \in,\ (E_\nu : \nu \leq \alpha))$$

which "look like" one of the W_ξ.

A *linear iteration* of a premouse $\mathcal{N}$ is a commuting system

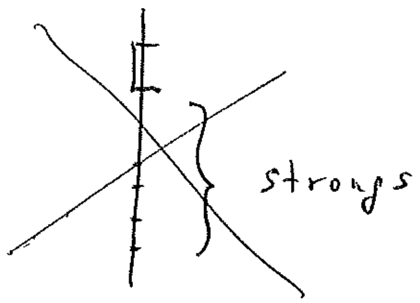
$$(\mathcal{M}_\alpha,\ \pi_{\alpha\beta} : \alpha \leq \beta < \gamma),$$

where $\mathcal{M}_0 = \mathcal{N}$, $\pi_{\alpha\alpha} = \mathrm{id}$,

$\mathcal{M}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}_\alpha^*;\ E)$ for some E from the extended sequence of $\mathcal{M}_\alpha$ (where, as above, $\mathcal{M}_\alpha^*$ is the longest initial segment of $\mathcal{M}_\alpha$ s.t. E measures all the subsets of $\mathrm{crit}(E)$ in $\mathcal{M}_\alpha^*$),

$\pi_{\alpha\alpha+1}$ is the ultrapower map, and direct limits are taken at limit stages.

A premouse W is said to be <u>below</u> $0^{\P}$ (0– <u>handgrenade</u>) iff there is no extender E on the sequence of W s.t. in W, crit(E) is a limit of strong cardinals.



A premouse W is <u>linearily</u> <u>iterable</u> iff all the models in any linear iteration of W are well-founded (and may hence be taken as being transitive).

Linearily iterable premice below $0^{\P}$ may be successfully compared in much the same way as any two iterable premice which "look like" $0^{\#}$ can be compared.

<u>Lemma</u> (Dodd, Jensen, Mitchell, Sch).

Let W, W' be linearily iterable premice which are below $O^{\P}$. There are then linear iterations

$$(M_\alpha, \pi_{\alpha\beta} : \alpha \leq \beta \leq \gamma) \quad \text{and}$$
$$(M'_\alpha, \pi'_{\alpha\beta} : \alpha \leq \beta \leq \gamma')$$

of W, W', respectively, s.t. M_γ is an initial segment of $M_{\gamma'}$, or vice versa.

<u>Lemma</u> (folklore among inner model theorists)

The models W_ξ from the K^c construction are all linearily iterable.

Let us sketch the proof of this.

Let W_ξ be given. By reflection, it suffices to show that if

$$\bar{W} \longrightarrow W_\xi$$

is elementary, and if

$$(\mathcal{M}_\alpha, \pi_{\alpha\beta} : \alpha \leq \beta < \gamma)$$

is a linear iteration of $\bar{W}$ with $\gamma < \omega_1$, then all the $\mathcal{M}_\alpha$ are transitive.

In fact, for each $\alpha < \gamma$ there is a map $\sigma_\alpha : \mathcal{M}_\alpha \longrightarrow W_\xi$ (or $\sigma_\alpha : \mathcal{M}_\alpha \to W_{\bar{\xi}}$, some $\bar{\xi} < \xi$). The maps σ_α commute with the maps $\pi_{\alpha\beta}$.

$$\begin{tikzcd}
\mathcal{M}_{\alpha+1} \arrow[ddr, dashed, "\sigma_{\alpha+1}"] & \\
\mathcal{M}_\alpha \arrow[u, "\pi_{\alpha\alpha+1}"] \arrow[dr, "\sigma_\alpha"'] & \\
\bar{W} \arrow[r] & W_\xi
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{M}_{\alpha+1} \arrow[ddr, dashed, "\sigma_{\alpha+1}"] & \\
\mathcal{M}_{\alpha} \arrow[u, "\pi_{\alpha\alpha+1}"] \arrow[dr, "\sigma_{\alpha}"'] & \\
\widetilde{W} \arrow[r] & W_{\xi}
\end{tikzcd}
$$

Say $\sigma_\alpha : \mathcal{M}_\alpha \longrightarrow W_\xi$ has been constructed,

and $\mathcal{M}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}_\alpha ; E)$.

As $\sigma_\alpha(E)$ is certified, it is ω-complete.

We'll have that every element of $\mathcal{M}_{\alpha+1}$ is of the form $\tilde{E}(f)(\eta) = \pi_{\alpha\alpha+1}(f)(\eta)$, where $\eta < \nu =$ the index of E in $\mathcal{M}_\alpha$.

By the ω-completeness of $\sigma_\alpha(E)$, there is a map

$$\Phi : \eta \longmapsto \Phi(\eta) < \mathrm{crit}(\sigma_\alpha(E)) = \sigma_\alpha(\mathrm{crit}(E))$$

s.t.

$$\tilde{E}(f)(\eta) = \pi_{\alpha\alpha+1}(f)(\eta) \longmapsto \sigma_\alpha(f)(\Phi(\eta))$$

is a map from $\mathcal{M}_{\alpha+1}$ to W_ξ. Obviously $\sigma_{\alpha+1} \circ \pi_{\alpha\alpha+1} = \sigma_\alpha$.

Above $0^{\P}$, the coiterability lemma of Dodd, Jensen, Mitchell, Sch breaks down and we need iteration trees rather than linear iterations in order to compare premice.

Let γ be an ordinal. For our purposes, a tree on γ is a tree order $<_T$ on γ which respects the natural order on γ s.t. if $\lambda < \gamma$ is a limit ordinal, then

$$\{\alpha : \alpha <_T \lambda\} \text{ is cofinal in } \lambda.$$

Now let W be a premouse, and let $<_T$ be a tree on γ in the above sense.

An <u>iteration</u> of $\mathcal{N}$ with tree structure $<_T$ is a commuting system

$$(\mathcal{M}_\alpha, \pi_{\alpha\beta} : \alpha \leq_T \beta < \gamma),$$

where $\mathcal{M}_0 = \mathcal{N}$, $\pi_{\alpha\alpha} = \mathrm{id}$,

$\mathcal{M}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}^*_\delta ; E)$ for some E from the extender sequence of $\mathcal{M}_\alpha$ (sic!), where $\delta =$ the predecessor of $\alpha+1$ in $<_T$ (and again, $\mathcal{M}^*_\delta$ is the longest initial segment of $\mathcal{M}_\delta$ s.t. E measures all the subsets of $\mathrm{crit}(E)$ in $\mathcal{M}^*_\delta$),

$\pi_{\delta\,\alpha+1}$ is the ultrapower map, and direct limits are taken at limit stages.

Iterability now becomes more subtle.

<u>Definition</u>. Let $\mathcal{N}$ be a premouse, and let γ be an ordinal.
We then say that $\mathcal{N}$ is γ-<u>iterable</u> iff there is a function Σ (an <u>iteration strategy</u>) s.t. the following holds true.
Let $(\mathcal{M}_\alpha, \pi_{\alpha\beta} : \alpha \leq_T \beta < \bar{\gamma})$ be an iteration of $\mathcal{N}$, where $\bar{\gamma} < \gamma$.
Then either:

$\bar{\gamma}$ is a successor ordinal and all the $\mathcal{M}_\alpha$ are transitive,

or else

there is a maximal branch b thru T given by Σ s.t. $\operatorname{dir\,lim}_{\alpha \leq_T \beta \in b} (\mathcal{M}_\alpha, \pi_{\alpha\beta})$ is transitive;

$b = \Sigma(\mathcal{M}_\alpha, \pi_{\alpha\beta} : \alpha \leq_T \beta < \sup(b))$.

In other words, there is a strategy Σ s.t. if

$$(\mathcal{M}_\alpha, \pi_{\alpha\beta} : \alpha \leq_T \beta < \lambda)$$

is already according to Σ, λ a limit ordinal $< \bar{\gamma}$, then Σ picks a cofinal branch b thru $<_T$ s.t.

$$\operatorname*{dir\,lim}_{\alpha \leq_T \beta \in b} (\mathcal{M}_\alpha, \pi_{\alpha\beta})$$

is transitive, and every iteration according to Σ of length $< \gamma$ only has transitive models.

The reflection argument as in the case of premice $\mathcal{W}$ below $0^{\P}$ which shows "if $\bar{\mathcal{W}} \longrightarrow \mathcal{W}$, $\bar{\mathcal{W}}$ ctble., is ctbly. iterable, then $\mathcal{W}$ is iterable" breaks down, though.

<u>Definition.</u> Let $\mathcal{W}$ be a premouse.
$\mathcal{W}$ is called <u>countably iterable</u> iff whenever $\pi : \bar{\mathcal{W}} \longrightarrow \mathcal{W}$, where $\bar{\mathcal{W}}$ is countable, then $\bar{\mathcal{W}}$ is $\omega_1 + 1$ iterable.
Countably iterable premice are also called

<u>mice</u>.

<u>Definition.</u> Let $\mathcal{W}$ be a premouse.
$\mathcal{W}$ is called <u>fully iterable</u> iff $\mathcal{W}$ is γ-iterable for all ordinals γ.

We'll late see an exaple of a premouse which (in some universe) is countably itrable but not fully itrable.

A premouse $\mathcal{W}$ is called <u>domestic</u> iff there is no extender E on the sequence of $\mathcal{W}$ s.t. in $\mathcal{W}$, $crit(E)$ is a limit of strong cardinals as well as of Woodin cardinals.

<u>Thm.</u> (Steel, Andretta–Neeman–Steel, Mitchell–Sch).

Suppose that all the models $\mathcal{W}_\xi$ from the K^c construction are domestic. Then every $\mathcal{W}_\xi$ is countably iterable.

The proof shows that if $\bar{\mathcal{W}} \longrightarrow \mathcal{W}_\xi$, where $\bar{\mathcal{W}}$ is countable, and if

$(\mathcal{M}_\alpha, \pi_{\alpha\beta} : \alpha \leq_T \beta < \gamma)$ is an iteration of $\bar{\mathcal{W}}$, $\gamma < \omega_1$, then the models $\mathcal{M}_\alpha$ can be "realized back into" $\mathcal{W}_{\bar{\xi}}$, $\bar{\xi} \leq \xi$:

$$\begin{tikzcd}
\mathcal{M}_{\alpha+1} \arrow[drr, dashed] & & \\
\mathcal{M}_\delta \arrow[u, bend left, "\pi_{\delta\alpha+1}"] \arrow[rr, "\sigma_\delta"'] & & \mathcal{W}_\xi \\
\bar{\mathcal{W}} \arrow[urr] & &
\end{tikzcd}$$

It is much more difficult to pass the limit stages of this construction, though.

As to the questions asked in the beginning of this talk:

(1) the proof of the countable iterability of $\mathcal{W}_\xi$ also shows the countable iterability of any bicephalus

$$(|\mathcal{M}_\xi|; \in, \vec{E}, F_0, F_1),$$

where F_0, F_1 are candidates for the next extender. This gives $F_0 = F_1$.

(2) + (3) Countable iterability shows that the fine structure theory of the $\mathcal{N}_\xi$'s works out. In particular, given any α, the $\mathcal{N}_\xi$'s agree on $\mathcal{N}_\xi \restriction \alpha$ from some point ξ_α onward. Therefore, $\mathcal{N}_{OR}$ is a class-sized model which we call

$$K^c.$$

The fine structure theory also allows us to analyze K^c. In particular:

<u>Theorem</u>. (Schimmerling–Zeman) Suppose all the models from the K^c construction are countably iterable. If κ is not subcompact in K^c, then $\square_\kappa$ holds in K^c.

<u>Open problem</u> :

Is there some κ s.t.

$$\kappa^{+K^c} = \kappa^+ .$$

If κ is measurable, then K^c correctly computes the successor of κ. The real problem is to show the existence of some such κ in ZFC + "all models from the K^c construction are countably iterable" + "K^c does not have a superstrong cardinal".

If $\kappa^{+K^c} = \kappa^+$, then the failure of $\square_\kappa$ in V would imply that κ is subcompact in K^c.

Whereas K^c may not be used directly, we may exploit the technique of stacking mice over $K^c|\kappa$, where κ is a regular cardinal (in V).

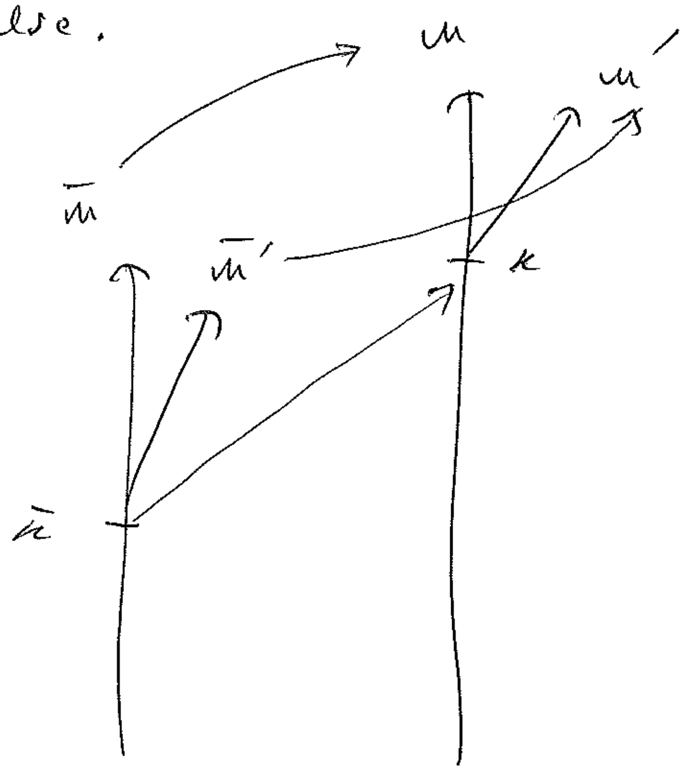
Fix κ, a regular cardinal s.t. $\alpha^{\aleph_0} < \kappa$ for all $\alpha < \kappa$. Consider

$\mathcal{S} = \{ \mathcal{M} :$ $\mathcal{M}$ is a mouse which end-extends $K^c|\kappa$,
$\mathrm{Hull}^{\mathcal{M}}(\kappa \cup \{p\}) = \mathcal{M}$, some p, where the hull is fine structurally,
but $\mathcal{M}$ does <u>not</u> define a new bounded subset of $\kappa \}$.

<u>Claim</u>. If $\mathcal{M}, \mathcal{M}' \in \mathcal{S}$, then $\mathcal{M}$ is an initial segment of $\mathcal{M}'$ or vice versa.

The proof of the claim is an application of fine structure theory, in particular of the condensation lemma.

Suppose $\mathcal{M}$ and $\mathcal{M}'$ witness the claim to be false.



Let $\pi : H \longrightarrow H_{\kappa^{++}}$, where H is transitive, $\mathrm{crit}(\pi) = \pi^{-1}(\kappa) = \bar{\kappa}$. By the condensation lemma, $\bar{\mathcal{M}} \lhd \mathcal{M}$ and $\bar{\mathcal{M}}' \lhd \mathcal{M}'$. Hence $\bar{\mathcal{M}} \lhd K^c$, $\bar{\mathcal{M}}' \lhd K^c$. So $\bar{\mathcal{M}}, \bar{\mathcal{M}}'$ are comparable; so $\mathcal{M}, \mathcal{M}'$ are comparable.

We may now let

$$S(K^c|\kappa) = \text{the union of all the mice in } \mathcal{S}.$$

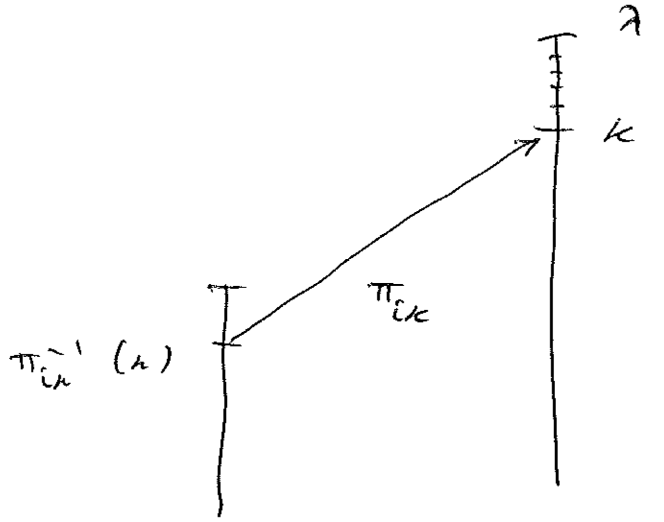
Write $S = S(K^c|\kappa)$. $S \models ZFC^-$ with largest cardinal κ. Set λ = the ordinal height of S.

<u>Theorem.</u> (Jensen, Schimmerling, Sch, Steel)

If $\kappa \geq \aleph_3$, then $cf(\lambda) \geq \kappa$.

The argument is reminiscent of the proof of covering for L.

Deny.



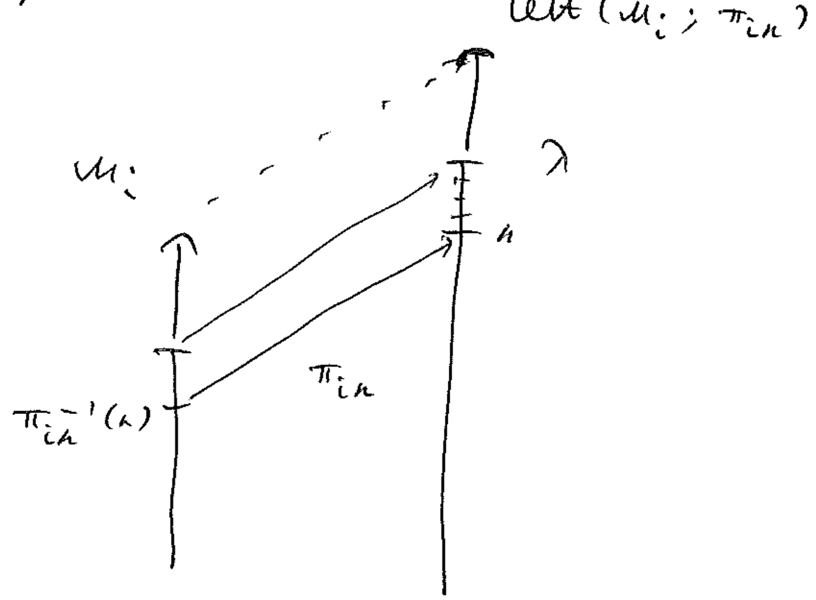
Let $X \subset \lambda$ witness $cf(\lambda) < \kappa$. Pick

$$(H_i, \pi_{ij} : i \le j \le \kappa)$$

commuting s.t. all H_i are transitive, $\overline{\overline{H_i}} < \kappa$ for $i < \kappa$, $crit(\pi_{i\kappa}) = \pi_{i\kappa}^{-1}(\kappa)$ for $i < \kappa$, $H_\kappa = S$, etc.

For a typical i, there is some $\mathcal{M}_i \trianglelefteq K^c$ which end-extends H_i over which a new subset of $\pi_{i\kappa}^{-1}(\kappa)$ is definable.

(Reason: Otherwise we could use $\pi_{i\kappa}$ to derive an extender which we forgot to put onto the K^c sequence. Notice that for a typical i, $\pi_{i\kappa}$ will be certified!)



For such a typical i, we plan to use

$$\mathrm{Ult}(\mathcal{M}_i\,;\,\pi_{i\kappa})$$

to argue that we forgot to add a mouse to the stack, More precisely, let

$$\tilde{\pi} : \mathcal{M}_i \longrightarrow W_i$$

be the natural extension of

$$\pi_{i\kappa} : H_i \longrightarrow H$$

(which is formed in much the same way as ultrapowers in the iteration of premice).

By the choice of X, W_i end-extends H.

Problem: W_i might define a new <u>bounded</u> subset of κ.

But this problem (and all the others!) can be solved to give a contradiction.

We arrive at :

<u>theorem</u>. (Jensen, Schimmerling, Sch, Steel)

If the Proper Forcing Axiom PFA holds, then there is a non-domestic premouse.

Proof sketch. Work with $\kappa = \aleph_3$.

If $\lambda =$ ordinal height of $S = S(K^c|\kappa) = \aleph_4$, we have $\square_\kappa$ using the $\square_\kappa$-sequence of S.

O.w. $cf(\lambda) = \omega_3$.

But then we can show that the $\square_\kappa$-sequence of S cannot be threaded.

In both cases, we get a contradiction with PFA, using a result of Todorcevic. $\dashv$

It remains to answer (4):

Can we prove covering for K^c?

Many arguments in inner model theory need a model which has stronger properties than K^c has: K^c is just countably iterable, whereas we often need full iterability to prove stronger results.

An example will be given next time. We'll then see how we may arrange working in local universes in which K^c is fully iterable. This will lead to the <u>core model induction</u>.