III.

Last time: We saw that the models W_ξ from the K^c construction are countably iterable as long as they are all domestic.

It was also said that we may not use a reflection argument to see that they are in fact fully iterable; actually, there are counterexamples.

Today, I first want to give an example of an application where full iterability would be needed. We'll then study the problem of the full iterability of K^c which will lead to the core model induction technique.

the pcf "conjecture" states that for a set a of regular cardinals,

$$\overline{\overline{pcf(a)}} = \overline{\overline{a}} .$$

It has to be wrong if $2^{\aleph_0} < \aleph_\omega$, but $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$.

We get models with Woodin cardinals from this hypothesis, which is not known to be consistent.

However, we also get models with Woodin cardinals from a hypothesis which Gitik has shown to be consistent and which is related to the pcf "conjecture."

<u>Theorem</u> (Gitik, Sch, Shelah)

Let κ be a singular cardinal of uncountable cofinality. Suppose

$$\{\alpha < \kappa : 2^{\alpha} = \alpha^{+}\}$$

to be stationary and co stationary.
Then for every $n < \omega$, there is an inner model with n Woodin cardinals.

I do not want to sketch the proof of this theorem, but I want to show you an aspect of the proof in order to convince you that full iterability of inner models is an issue.

The above hypothesis formulates a strong version of the failure of SCH.

The hypothesis of the theorem gives many increasing sequences

$$(\kappa_i : i < \omega)$$

of singular cardinals below κ s.t.

$$cf\left(\prod \kappa_i^+\right) > (\sup_i \kappa_i)^+ = \lambda^+.$$

The plan is to show that for $W = K^c$ (or $W =$ a better model than K^c) s.t.

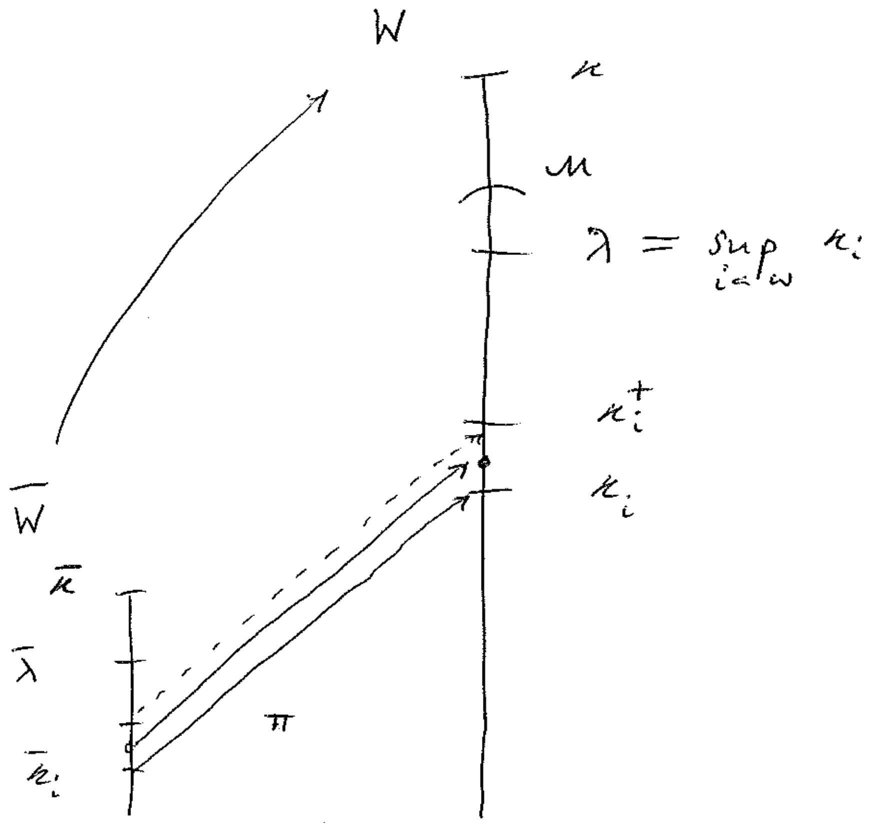
$W \models GCH$,

$$\{ f \restriction \{\kappa_i : i < \omega\} : \begin{array}{l} f \in W \\ f : \lambda \to \lambda \end{array} \}$$

is cofinal in $\prod \kappa_i^+$.

This will certainly yield a contradiction.

The key idea is to use a <u>covering argument</u>.



Let $f \in \prod_{i<\omega} \kappa_i^+$; $f : \omega \to \lambda$, $f(i) < \kappa_i^+$

f.a. $i<\omega$.

Pick $\pi : \bar{W} \longrightarrow W$ s.t. $\bar{W}$ is transitive,

$\mathrm{Card}(\bar{W}) = \aleph_1$, $f(i) \in \mathrm{ran}(\pi)$ f.a. $i<\omega$.

The plan is to argue that there be some $\mathcal{M} \lhd W$ s.t. for all but finitely many $i<\omega$,

$$\mathrm{ran}(\pi) \cap \kappa_i^+ \subset \mathrm{Hull}^{\mathcal{M}}(\kappa_i \cup \{p\}),$$

some fixed $p \in \mathcal{M}$.

We may then set

$$\tilde{f}(\xi) = \sup \left(\mathrm{Hull}^{\mathcal{M}} (\xi \cup \{p\}) \right) \cap \xi^{+K},$$

where $\xi < \lambda$.

Then $\tilde{f} : \lambda \to \lambda$, $\tilde{f} \in W$, and because

$$\kappa_i^{+W} = \kappa_i^+$$

f.a. $i < \omega$ (as all the κ_i are singular) and

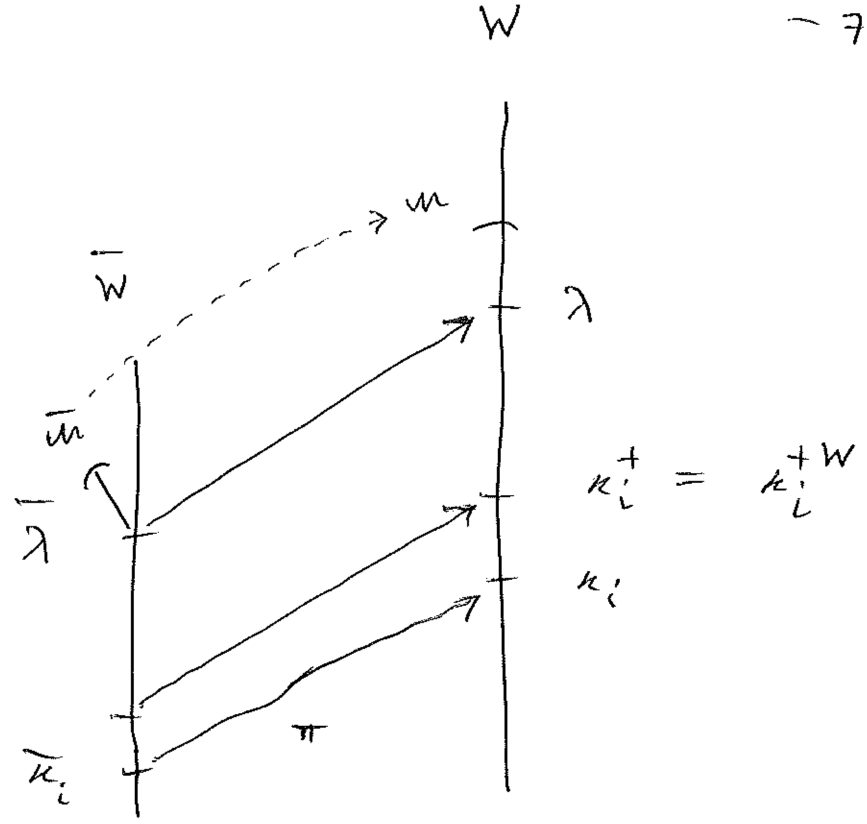
$$f(i) \in \mathrm{ran}(\pi) \cap \kappa_i^+ \subset \mathrm{Hull}^{\mathcal{M}} (\kappa_i \cup \{p\}),$$

we get that

$$f(i) < \tilde{f}(\kappa_i).$$

I.e., $\tilde{f} \upharpoonright \{\kappa_i : i < \omega\}$ majorizes f, and $\tilde{f} \in W$. $\tilde{f}$ is thus as desired.

Where do we get such an $\mathcal{M}$ from?



The plan for this is:

- Coiterate $\bar W$, W.
- Show that (π may have been chosen in such a way that) $\bar W$ does not move in the coiteration.
- The coiteration produces an $\bar{\mathcal{M}}$ s.t.

$$\pi^{-1}(\kappa_i^+) \subset \mathrm{Hull}^{\bar{\mathcal{M}}}(\bar\kappa_i \cup \{\bar p\}), \text{ some } \bar p.$$

- Then, setting $\mathcal{M} = \mathrm{Ult}(\bar{\mathcal{M}}; \pi \restriction \bar\lambda)$,

$$\mathrm{ran}(\pi) \cap \kappa_i^+ \subset \mathrm{Hull}^{\mathcal{M}}(\kappa_i \cup \{p\}),$$

where $p = \pi_{\bar{\mathcal{M}}\mathcal{M}}(\bar p)$.

Point is: We obviously need more than countable iterability of W to show that this works.

We in fact need the full iterability of W!

On the other hand, the iterability proof for (the models from the) K^c (construction) really just produces ctble. iterability.

We have to use a <u>reflection</u> <u>argument</u> to show

ctble. iterability $\Longrightarrow$ full iterability.

In order for this reflection argument to work out, we need that V is closed under operators which certify branches thru iteration trees.

Let the premouse $\mathcal{M}$ be ctbly. iterable. How would you try proving that $\mathcal{M}$ is fully iterable?

(1) We need a candidate for a full iteration strategy for $\mathcal{M}$. Call it Σ.

(2) We need to argue: if the iteration

$$\tilde{\mathcal{J}} = (\mathcal{M}_\alpha, \pi_{\beta\alpha} : \beta \leq_T \alpha < \gamma)$$

is according to Σ, then all the models from $\mathcal{J}$ are transitive, and if γ is a limit ordinal, then $\Sigma(\mathcal{J}) \downarrow$.

We need to reflect a potential failure of (2) down into H_{ω_1}.

$$\begin{tikzcd} \bar{\mathcal{T}} & & \mathcal{T} \\ \bar{\mathcal{M}} \arrow[u, squiggly] \arrow[rr, "\sigma"'] & & \mathcal{M} \arrow[u, squiggly] \end{tikzcd}$$

Pick $\sigma : H \longrightarrow V$, H ctble. and transitive.

Let $\bar{\mathcal{M}}, \bar{\mathcal{T}} = \sigma^{-1}(\mathcal{M}, \mathcal{T})$.

$$\bar{\mathcal{T}} = (\bar{\mathcal{M}}_\alpha, \bar{\pi}_{\beta\alpha} : \beta \leq_{\bar{T}} \alpha < \bar{\gamma})$$

is a ctble. iteration of $\bar{\mathcal{M}}$.

Suppose $\mathcal{T}$ is according to Σ, all the models are transitive, γ (and hence $\bar{\gamma}$) is a limit ordinal, and we search for a cofinal branch thru $\mathcal{T}$ (which is according to Σ).

Let $\bar{\Sigma}$ be an iteration strategy for $\bar{\mathcal{M}}$ w.r.t. countable iterations of $\bar{\mathcal{M}}$. So

$$\bar{\Sigma}(\bar{\mathcal{T}}) \downarrow, \text{ say } = b.$$

Say there is an initial segment

$$Q \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{I}}} = \text{the direct limit model according to } b$$

which can be identified in H, i.e., is an element of H and is definable in H.

Then by absoluteness, for the right θ,

$$H^{\mathrm{Col}(\omega,\theta)} \models \text{" there is a cofinal branch } b' \text{ thru } \bar{\mathcal{I}} \text{ s.t. } Q \trianglelefteq \mathcal{M}_{b'}^{\bar{\mathcal{I}}} \text{ ,"}$$

and if Q identifies b, then $b' = b \in H$ by homogeneity and b is definable in H via Q.

But then $\sigma(b)$ is a perfect candidate for $\Sigma(\bar{\mathcal{I}})$.

<u>Example:</u> If there is no inner model with a Woodin cardinal and $\mathcal{M}$ has no definable Woodin cardinal, then this argument works with

Q = the least initial segment of $L[\mathcal{M}(\mathcal{I})]$ which kills the Woodinness of $\delta(\mathcal{I})$.

(Here, $\delta(\mathcal{I})$ = sup of the indices of the extenders used in $\mathcal{I}$; $\mathcal{M}(\mathcal{I})$ = the "common part model" of $\mathcal{I}$; $\mathcal{N} \trianglelefteq \mathcal{M}(\mathcal{I})$ iff $\mathcal{N} \trianglelefteq \mathcal{M}_\alpha$ for a tail end of α's, where $\mathcal{M}_\alpha$ is the α^{th} model from the iteration $\mathcal{I}$.)

On the other hand, under unfavorable circumstances, models with Woodin cardinals need not be fully iterable:

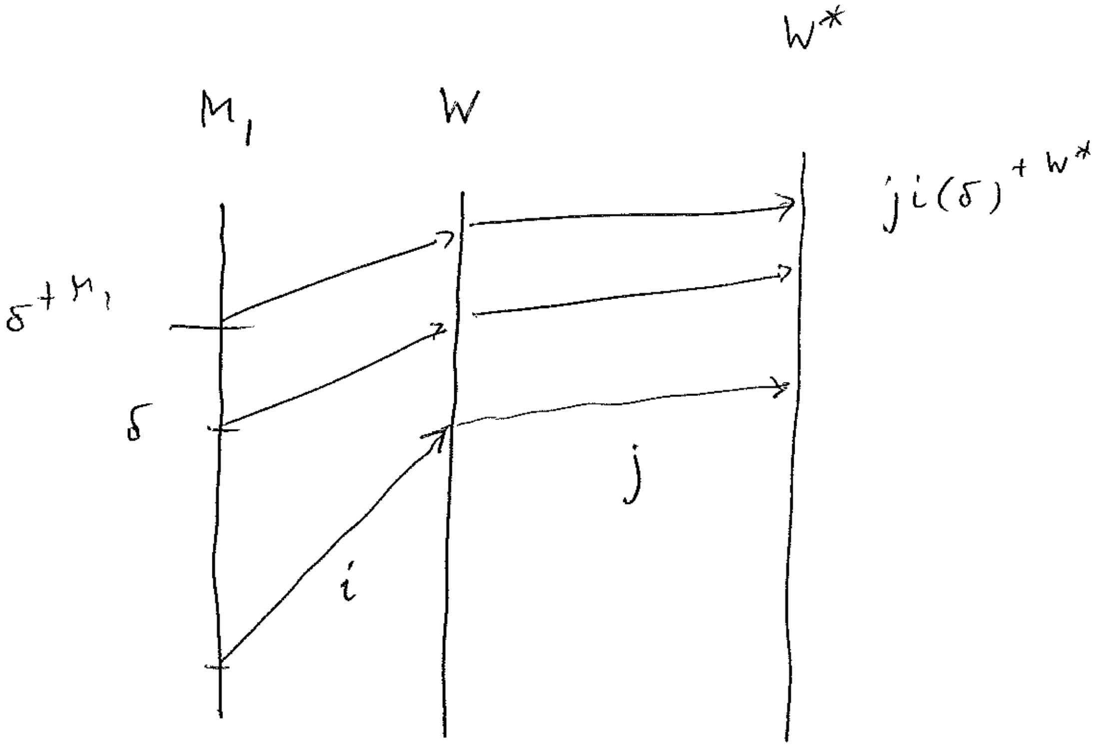
<u>Theorem</u> (Woodin). Let $\mathcal{M}$ be a fully iterable premouse, $\mathcal{M} \models$ "δ is a Woodin cardinal." There is then a poset $\mathbb{P} \in H^{\mathcal{M}}_{\delta^+}$ which has the δ-c.c. in $\mathcal{M}$ s.t. for every set A of ordinals whatsoever there is some iterate

$$i : \mathcal{M} \longrightarrow \mathcal{M}^*$$

s.t. A is $i(\mathbb{P})$-generic over $\mathcal{M}^*$.

Now let $M_1 = L[E]$ be the least premouse with a Woodin cardinal, δ. Basically, $E \subset \delta$. Suppose that

$$M_1 \models \text{"I'm fully iterable."}$$



Let W = the iterate of M_1 obtained by hitting the least measure of M_1 (and its images) δ^+ times, and let W^* be a further iterate s.t. E is generic over W^*. Then

$$W^*[E] = L[E] = M_1 .$$

ji "$\in$" M_1, so $ji \restriction \delta^+$ witnesses that in M_1,

$$cf(ji(\delta^{+M_1})) = cf(ji(\delta)^{+W^*})$$
$$= cf(ji(\delta)^{+M_1}) = \delta^+.$$

Contradiction!

There is hence nothing that might guarantee in general that K^c, albeit always being countably iterable, is fully iterable.

As in the example of M_1, it might just be that V is not saturated by the relevant Q-structures which identify cofinal branches thru iterations of K^c.

The idea of the <u>core model induction</u>, first developped by H. Woodin and later extended by J. Steel and others, is to inductively show V is closed under the relevant Q-structures and always work in local universes in which the K^c produced there is either fully iterable or provides the "next Q-structure."

Let us discuss this in the case of the above example in which κ is a singular cardinal, $\mathrm{cf}(\kappa) > \omega$, and

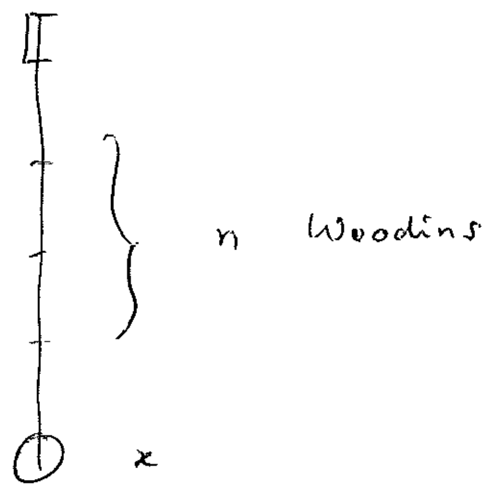
$$\{\alpha < \kappa : 2^{\alpha} = \alpha^{+}\}$$

is stationary and costationary.

We may then first show that every set in H_κ has a $\#$.

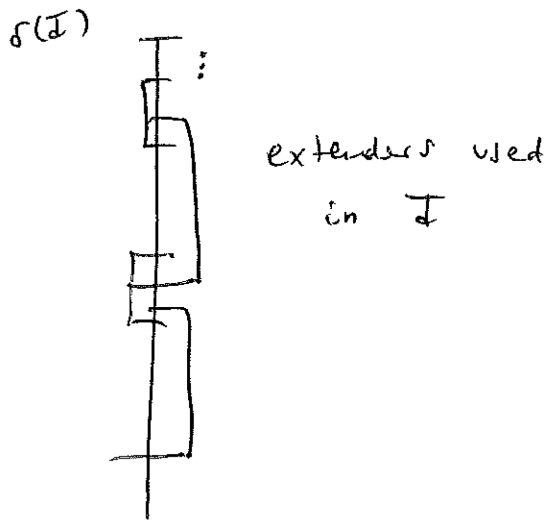
Now suppose that for every set x in H_κ, $M_n^{\#}(x)$ exists, but $M_{n+1}^{\#}(x_0)$ does not exist, some $x_0 \in H_\kappa$. Say $x_0 = \emptyset$.

Here, $M_n^{\#}(x) =$ the least premouse over x which has a measure above n Woodin cardinals and which is ctbly. iterable.



In this situation, let $\mathcal{I}$ be an iteration of K^c, say, where $\mathcal{I}$ has limit length $< \kappa$, and $\mathcal{I}$ lives on $K^c | \lambda$, some $\lambda < \kappa$.

Let $\mathcal{M}(\mathcal{I})$ be the common part model of $\mathcal{I}$, and let $\delta(\mathcal{I})$ be its height.



Then (an initial segment of) $M_n^{\#}(\mathcal{M}(\mathcal{I}))$ will serve as the Q-structure which identifies the correct branch thru $\mathcal{I}$.

Uses:

<u>Theorem.</u> (Martin, Steel) If $b \neq c$ are cofinal branches thru $\mathcal{I}$, then $\delta(\mathcal{I})$ is Woodin in $wfp(\mathcal{M}_b^{\mathcal{I}}) \cap wfp(\mathcal{M}_c^{\mathcal{I}})$.

The reflection argument from above then thus shows that $K^c|\kappa$ is κ-iterable.

We may then isolate a model W, namely the true <u>core</u> <u>model</u> K of height κ, for which the covering argument which we discussed above can be made work.

We'll have

$$K \cong X \prec K^c|\kappa$$

for an appropriate hull X. K will have the following property:

If
$$\sigma : W \longrightarrow K,$$

then either W looses the coiteration against K (i.e., is strictly weaker than K), or else $W = K$. (<u>Rigidity</u>)

Other properties of K :

Forcing absoluteness : $K^{V^{\mathbb{P}}} = K$ for all $\mathbb{P} \in H_\kappa$.

Weak covering : $cf(\lambda^{+K}) \geq \overline{\overline{\lambda}}$ whenever $\aleph_2 \leq \lambda < \kappa$.

This is a theorem of Mitchell, Schimmerling, Steel.

Local definability : $K|\lambda$ may be defined inside H_λ, where $2^{\aleph_0} < \lambda < \kappa$.

K inherits the full iterability from K^c.

Thrn results of Martin, Steel, and Woodin, the above argument shows <u>Projective Determinacy</u>, i.e., that all sets of reals which are in $J_2(\mathbb{R})$ are determined.

[$J_1(\mathbb{R}) = V_{\omega+1}$, etc.]

The core model induction now uses $L(\mathbb{R})$ as its guide in that:

We show inductively that (an initial segment of) V is closed under mice which correspond to the determinacy of all sets of reals in $J_\alpha(\mathbb{R})$, $\alpha \geq 2$.

Either the "next" mouse with a Woodin cardinal exists, or else we may isolate K to derive a contradiction.

The mouse closure will serve as a basis for the models we are about to produce to have terms in them which capture a given set of reals of the next complexity class; we'll use:

<u>Definition.</u> Let M be a countable mouse with a Woodin cardinal, δ. Let $A \subset \mathbb{R}$, let $\tau \in M^{\mathrm{Col}(\omega,\delta)}$, and let Σ be the iteration strategy for M. We then say that τ, Σ <u>capture</u> A iff for all

$$i : M \longrightarrow M^* \qquad (M^* \text{ still cble.})$$

according to Σ and for all g $\mathrm{Col}(\omega, i(\delta))$-generic over M^*, $g \in V$,

$$A \cap M^*[g] = \tau^g .$$

In the above situation,

$$A = \bigcup \{ \tau^g : g \in V \text{ generic over a ctble. iterate } M^* \text{ of } M \}$$

<u>theorem.</u> (Neeman) Let M, A, τ, Σ be as above. Then A is determined.

The core model induction has various cases.

Notice:

" There is a set of reals which is not determined " is Σ_1,

if we count $\forall x \in \mathbb{R}$ and $\exists x \in \mathbb{R}$ as bounded quantification.

Therefore, if α is least s.t. $J_\alpha \not\models AD$

(AD = the axiom of determinacy), then

α begins a Σ_1-gap :

Definition. Let $\alpha \leq \beta$. Then $[\alpha, \beta]$ is a Σ_1-gap (in $L(\mathbb{R})$) iff

- $J_\alpha(\mathbb{R}) \prec^{\mathbb{R}}_{\Sigma_1} J_\beta(\mathbb{R})$
- $J_{\bar\alpha}(\mathbb{R}) \not\prec^{\mathbb{R}}_{\Sigma_1} J_\alpha(\mathbb{R})$ for all $\bar\alpha < \alpha$
- $J_\beta(\mathbb{R}) \not\prec^{\mathbb{R}}_{\Sigma_1} J_{\bar\beta}(\mathbb{R})$ for all $\bar\beta > \alpha$.

The Σ_1-gaps partition the class of all ordinals.

The core model induction works by induction on the gaps.

Main cases:

(1) α is inadmissible and the previous gap, if there is one, is not strong

(2) α ends a weak ~~proper~~ gap or it begins one, and there is a previous gap which is strong.

In the inadmissible gap case we can proceed as discussed above.

In the weak gap case we have to construct a new kind of premice, hybrids.

Say $[\beta, \alpha]$, $\beta < \alpha$, is the weak gap.
Let $m < \omega$ be least s.t. a new set of reals, A, is $\Sigma_m^{J_\alpha(\mathbb{R})}$-definable.
Then $A = \bigcup_{n<\omega} A_n$, where $A_n \in J_\alpha(\mathbb{R})$ $\forall n$.

The inductive hypothesis will give us a "suitable" premouse with an iteration strategy with condensation, i.e.

a cble. mouse W with an iteration strategy Σ, $W \models$ "δ is Woodin", and terms τ_n, $n < \omega$, s.t. τ_n, Σ capture A_n $\forall n$.

The hybrids look like ordinary mice except for that where we closed under rud before we will now in addition feed in information about how to iterate $\mathcal{N}$ according to Σ.

<u>Hybrid</u> <u>premouse</u>: $J_\gamma[\mathcal{N}, \vec{E}, \Sigma]$.

As Σ satisfies condensation, we may do a $K^{c,\Sigma}$ construction in much the same way as we did a K^c construction before.

Once we found a hybrid mouse with a Woodin cardinal which has an iteration strategy Γ which moves Σ correctly, we may use Neeman's theorem to deduce A is determined:

Let $M = J_\gamma[\mathcal{N}, \vec{E}, \Sigma]$ be a hybrid mouse with a Woodin cardinal, δ.

We may define a term $\tau \in M^{\mathrm{Col}(\omega,\delta)}$ in such a way that for $x \in \mathbb{R} \cap M^{\mathrm{Col}(\omega,\delta)}$,

$\Vdash x \in \tau$ iff

if x is made generic over an iterate of $\mathcal{N}$ using Σ, then x is in the interpretation of the image of one τ_n, new.

τ will then capture A.

We need that M be iterable in a way that Σ is moved correctly, and that Σ, as given to M, will extend to Σ, restricted to $M^{\mathrm{Col}(\omega,\delta)}$, in a definable way.

Applications of the core model induction :

Theorem (Woodin) If there is an ω_1-dense ideal on ω_1, then $AD^{L(\mathbb{R})}$ holds.

Theorem. (Steel) If PFA holds, then $AD^{L(\mathbb{R})}$ holds.

(The stacking technique today gives a stronger result, but it might be that an extension of the core model induction produces a stronger result than the stacking technique.)

Theorem (Busche, Schindler) If every uncountable cardinal is singular, then $AD^{L(\mathbb{R})}$ holds.

Extensions of the core model induction technique beyond $L(\mathbb{R})$:

<u>Theorem</u> (Ketchersid) Suppose CH + there is an ω_1-dense ideal on ω_1 + ε. There is then a model of $AD + \theta_0 < \theta$ of the form $L(\mathbb{R}, A)$, some $A \subset \mathbb{R}$.

The set A in this theorem is actually an iteration strategy for a "full" mouse producing $HOD | \theta$ of the maximal model of $AD + \theta_0 = \theta$.

More generally:

<u>Theorem.</u> (Sargsyan) Suppose CH + there is an ω_1-dense ideal on ω_1 + ε. There is then a model of $AD_{\mathbb{R}}$ + θ is regular.

By work of Woodin, this gives an equiconsistency.

The proof of the Ketchersid-Sargsyan result uses an extension of the core model induction technique beyond $L(\mathbb{R})$.

Given a model $L(\mathbb{R}, \Gamma)$ of $AD + \theta_\alpha = \theta$, one starts out by analyzing its $HOD \restriction \theta$ and representing it as a direct limit of a countable hod-mouse $\mathcal{N}$. One then finds an iteration strategy Σ for $\mathcal{N}$ which cannot be in $L(\mathbb{R}, \Gamma)$; using condensation for Σ, one runs a core model induction to show AD in $L(\mathbb{R}, \Sigma)$. But $L(\mathbb{R}, \Gamma)$ was taken to be maximal, and therefore $AD + \theta_\alpha < \theta$ holds true in $L(\mathbb{R}, \Sigma)$.

Questions.

(1) Suppose κ is a limit cardinal with $\omega < cf(\kappa) < \kappa$, and

$$\{\alpha < \kappa : 2^{\alpha} = \alpha^{+}\}$$

is stationary and costationary in κ.

Does AD hold in $L(\mathbb{R})$?

Is there a model of AD + Θ regular ?

(2) Suppose that every uncountable cardinal is singular.

Is there a model of AD + Θ regular ?

How do you go beyond AD + Θ regular from these hypotheses ?

<u>Further questions</u>:

(3) Let κ be a singular strong limit cardinal, and suppose $\square_\kappa$ fails.

Is there a model of $AD + \theta$ regular ?

(4) Suppose PFA holds.

Is there a model of $AD + \theta$ regular ?

Is there an inner model with a supercompact cardinal ?

(5) Suppose κ is strongly compact.

Is there an inner model with a supercompact cardinal ?

(4) + (5) are certainly holy grails of inner model theory.